

Some Principles of Web Mercator Maps and their Computation

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Abstract

The Web Mercator projection is used widely for online interactive maps. Despite discussions of the appropriateness of a simplified spherical approach or fundamental reservations against the Mercator projection in general, its broad availability for everyone gives it a great significance. For example, OpenStreetMap provides free geographical data which is typically accessed by means of map tile servers which rely on the Web Mercator projection. In this paper, starting with the mathematical derivation of spherical Mercator projections, the error introduced by identifying ellipsoidal and spherical coordinates is analyzed, the map tile disassembling is described in full detail and map applications are given.

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1 Introduction

The Web Mercator (EPSG:3857) projection is used by OpenStreetMap, OsmAnd, Google Maps, Bing Maps, and many other services to provide zoomable *general purpose* maps, typically, but not necessarily for online use. According to Spatialreference.org 2020b, it is intended for visualization applications and it is not a recognized geodetic system. The reason is quite clear, because this projection uses data complying to the ellipsoidal coordinates for WGS 84 (World Geodetic System 1984) and develops the data on a sphere which is then projected to a planar map by a Mercator projection. This gives an error difference between the “real” Mercator projection from the WGS 84 ellipsoid and the spherical Web Mercator or Pseudo Mercator projection. The location error grows from the equator up to 50 km for the polar latitudes 85° S resp. 85° N, see Battersby, Finn, Usery, and Yamamoto 2014. Nevertheless, for *general purpose* maps, the visual difference is quite neglectable for the benefit of simplified computations. Details are discussed in Section 4. For small scale scientific maps there are accurate alternatives like GMT (Wessel, Luis, Uieda, Scharroo, Wobbe, Smith, and Tian 2019).

Besides the errors from the spherical simplifications, the Mercator projection has several questionable aspects, especially on a global scale where areas near the equator are depicted much smaller than areas nearer to the poles. This is also seen for single map tiles as is shown in Section 4.3. But, such aspects are not subject of examination here. The emphasis of this work is on mathematical detail for technical applications and algorithmic implementations.

In this paper, the mathematical basics of the spherical Mercator projection with loxodromes as starting point are described first. A loxodrome is a curve which crosses all meridians with a constant angle and is depicted as straight line on a Mercator map, see Figure 1. Next, the derived projection formulas are specifically adapted for web maps with a clipped geographic area. The projection image becomes a unit square $[0, 1]^2$. The disassembling of this unit square into map tiles used by tile servers is developed in full detail. Further, map drawing applications with map tiles are given according to different requirements. Additionally, the presentation of the Gudermann function, spherical loxodromic and orthodromic distances are discussed.

This paper provides the technical background and the mathematics for the algorithms used by the \LaTeX package mercatormap (Sturm 2020).

2 Spherical Mercator Projection

The Mercator projection for depicting the earth surface on a map conserves the angles with the meridians. Such, the loxodromes or rhumb lines become straight lines on the resulting map, see Figure 1.

The following considerations are based on a perfect sphere and do not respect a more precise ellipsoidal model for earth. Furthermore, it is sufficient to consider a unit sphere with radius 1.

2.1 Loxodrome Description

The radius vector \vec{r} of the sphere (position vector for the sphere points) can be given in dependency of a latitude angle $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and a longitude angle $\lambda \in \mathbb{R}$ as

$$\vec{r}(\varphi, \lambda) = \begin{pmatrix} \cos \lambda \cdot \cos \varphi \\ \sin \lambda \cdot \cos \varphi \\ \sin \varphi \end{pmatrix} \quad (1)$$

Of course, the longitude λ has a period of 2π and we allow $\lambda \in \mathbb{R}$ for now.

Orthogonal to the radius vector \vec{r} are the unit vector $\vec{\Lambda}$ showing in direction of a latitude circle and the unit vector $\vec{\Phi}$ showing in direction of a meridian:

$$\vec{\Lambda}(\varphi, \lambda) = \frac{1}{\cos \varphi} \cdot \frac{\partial \vec{r}}{\partial \lambda}(\varphi, \lambda) = \begin{pmatrix} -\sin \lambda \\ \cos \lambda \\ 0 \end{pmatrix} \quad (2)$$

$$\vec{\Phi}(\varphi, \lambda) = \frac{\partial \vec{r}}{\partial \varphi}(\varphi, \lambda) = \begin{pmatrix} -\cos \lambda \cdot \sin \varphi \\ -\sin \lambda \cdot \sin \varphi \\ \cos \varphi \end{pmatrix} \quad (3)$$

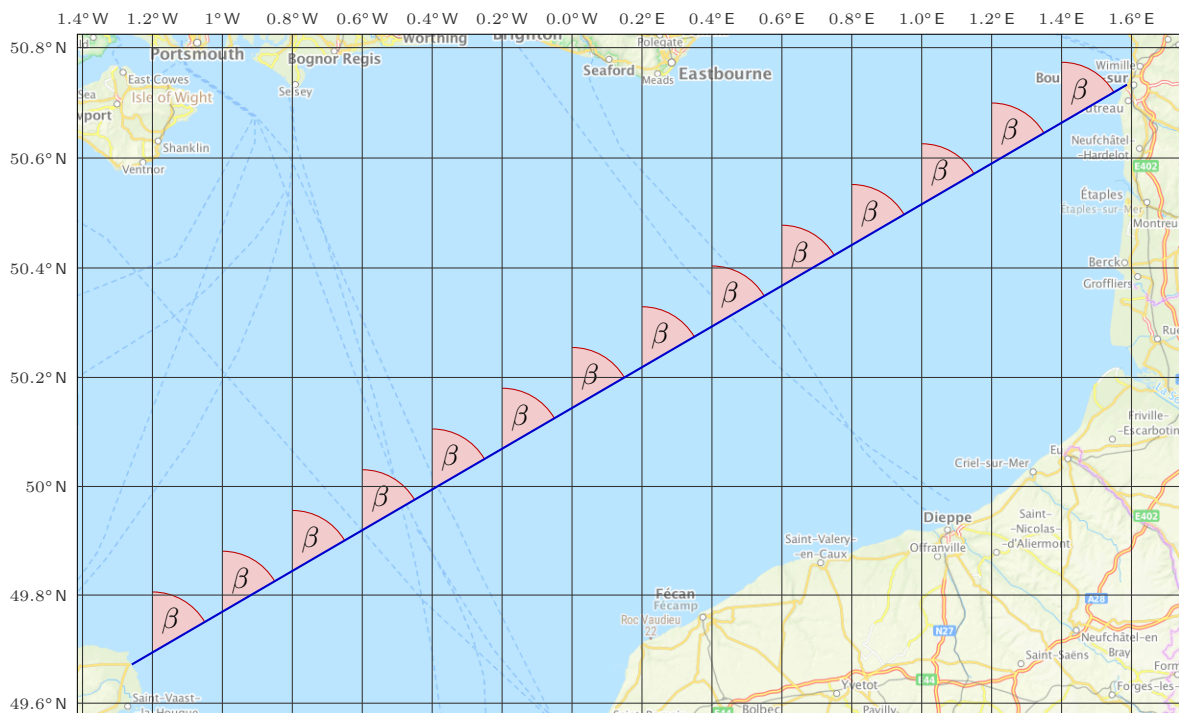


Figure 1: Course following a loxodrome. The heading angle between ship and meridians is always a constant β . On a nautical chart with Mercator projection this course is a straight line.

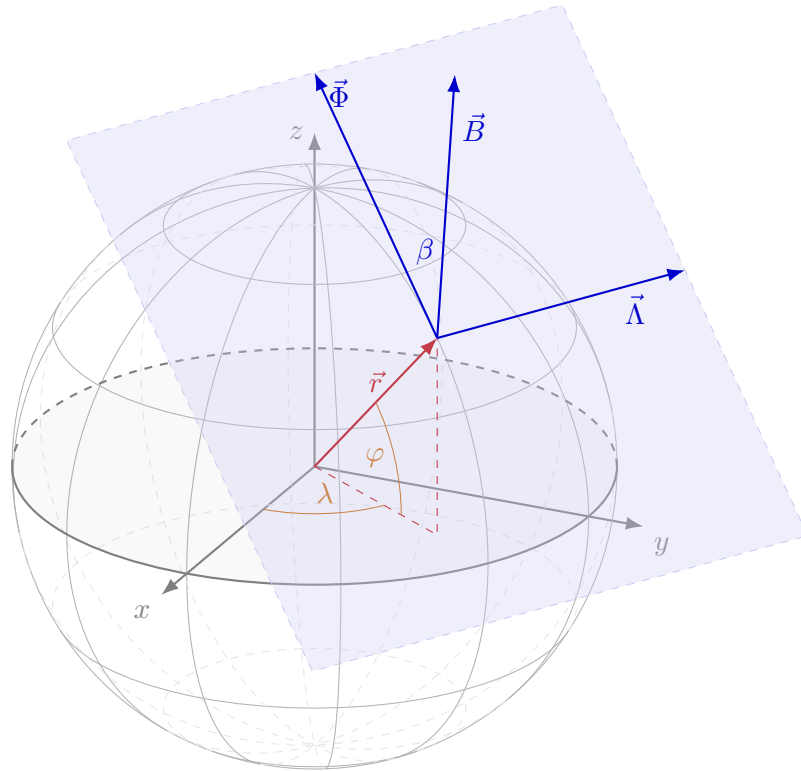


Figure 2: Tangent plane at a sphere point denoted by \vec{r} with latitude φ and longitude λ . The plane is spanned by $\vec{\Lambda}$ and $\vec{\Phi}$. \vec{B} is a vector in the plane with angle β to $\vec{\Phi}$.

For every (φ, λ) , the three vectors form an orthonormal system since we have

$$\langle \vec{r} | \vec{\Lambda} \rangle = \langle \vec{r} | \vec{\Phi} \rangle = \langle \vec{\Lambda} | \vec{\Phi} \rangle = 0.$$

The vectors $\vec{\Lambda}$ and $\vec{\Phi}$ span the tangent plane for \vec{r} , see Figure 2.

Now, consider another unit vector \vec{B} in this tangent plane with angle β to $\vec{\Phi}$ (meridian). This vector is given by

$$\vec{B}(\varphi, \lambda) = \cos \beta \cdot \vec{\Phi}(\varphi, \lambda) + \sin \beta \cdot \vec{\Lambda}(\varphi, \lambda), \tag{4}$$

because the scalar product of \vec{B} with $\vec{\Phi}$ is $\cos \beta$:

$$\langle \vec{B} | \vec{\Phi} \rangle = \cos \beta \cdot \langle \vec{\Phi} | \vec{\Phi} \rangle + \sin \beta \cdot \langle \vec{\Lambda} | \vec{\Phi} \rangle = \cos \beta.$$

A loxodrome $\vec{\gamma}$ with heading angle β and representation

$$\vec{\gamma}(s) = \vec{r}(\tilde{\varphi}(s), \tilde{\lambda}(s))$$

which is parameterized by arc length has to fulfill the differential equation system

$$\frac{d\vec{\gamma}}{ds}(s) = \vec{B}(\tilde{\varphi}(s), \tilde{\lambda}(s)) \tag{5}$$

Here, we deliberately ignore the second case $\frac{d\vec{\gamma}}{ds}(s) = -\vec{B}(\tilde{\varphi}(s), \tilde{\lambda}(s))$ which leads to loxodromes heading from north to south.

On the left hand side of the (5) it holds

$$\frac{d\vec{\gamma}}{ds}(s) = \frac{d}{ds}(\vec{r}(\tilde{\varphi}(s), \tilde{\lambda}(s))) = \frac{\partial \vec{r}}{\partial \varphi}(\tilde{\varphi}(s), \tilde{\lambda}(s)) \cdot \frac{d\tilde{\varphi}}{ds}(s) + \frac{\partial \vec{r}}{\partial \lambda}(\tilde{\varphi}(s), \tilde{\lambda}(s)) \cdot \frac{d\tilde{\lambda}}{ds}(s)$$

With (2), (3), and (4), (5) becomes

$$\begin{aligned} & \frac{d\tilde{\varphi}}{ds}(s) \cdot \vec{\Phi}(\tilde{\varphi}(s), \tilde{\lambda}(s)) + \frac{d\tilde{\lambda}}{ds}(s) \cdot \cos \tilde{\varphi}(s) \cdot \vec{\Lambda}(\tilde{\varphi}(s), \tilde{\lambda}(s)) \\ &= \cos \beta \cdot \vec{\Phi}(\tilde{\varphi}(s), \tilde{\lambda}(s)) + \sin \beta \cdot \vec{\Lambda}(\tilde{\varphi}(s), \tilde{\lambda}(s)) \end{aligned} \tag{6}$$

Following from the uniqueness of a basis representation or simply by scalar multiplication with $\vec{\Lambda}$ and $\vec{\Phi}$, (6) is equal to the following two equations:

$$\frac{d\tilde{\varphi}}{ds}(s) = \cos \beta. \tag{7}$$

$$\frac{d\tilde{\lambda}}{ds}(s) \cdot \cos \tilde{\varphi}(s) = \sin \beta, \tag{8}$$

Now, we change parametrization by arc length to parametrization by latitude using the substitution

$$\varphi = \tilde{\varphi}(s) \Leftrightarrow s = \tilde{\varphi}^{-1}(\varphi) = s(\varphi)$$

With (7), we have

$$1 = \frac{d(\tilde{\varphi} \circ \tilde{\varphi}^{-1})(\varphi)}{d\varphi} = \frac{d\tilde{\varphi}}{ds}(\tilde{\varphi}^{-1}(\varphi)) \cdot \frac{d\tilde{\varphi}^{-1}}{d\varphi}(\varphi) = \frac{d\tilde{\varphi}}{ds}(s) \cdot \frac{d\tilde{\varphi}^{-1}}{d\varphi}(\varphi) = \cos \beta \cdot \frac{d\tilde{\varphi}^{-1}}{d\varphi}(\varphi),$$

which gives the reciprocal of (7), namely

$$\frac{ds}{d\varphi}(\varphi) = \frac{d\tilde{\varphi}^{-1}}{d\varphi}(\varphi) = \frac{1}{\cos \beta} \tag{9}$$

With

$$\lambda(\varphi) := \tilde{\lambda}(s) = (\tilde{\lambda} \circ \tilde{\varphi}^{-1})(\varphi),$$

we get with (8) and (9):

$$\begin{aligned} \frac{d\lambda}{d\varphi}(\varphi) &= \frac{d(\tilde{\lambda} \circ \tilde{\varphi}^{-1})(\varphi)}{d\varphi} = \frac{d\tilde{\lambda}}{ds}(\tilde{\varphi}^{-1}(\varphi)) \cdot \frac{d\tilde{\varphi}^{-1}}{d\varphi}(\varphi) \\ &= \frac{d\tilde{\lambda}}{ds}(s) \cdot \frac{1}{\cos \beta} = \frac{\sin \beta}{\cos \tilde{\varphi}(s)} \cdot \frac{1}{\cos \beta} = \tan \beta \cdot \frac{1}{\cos \varphi}. \end{aligned} \tag{10}$$

With (10), a relationship between φ and λ is found. By adding a condition

$$\lambda(\varphi_0) = \lambda_0$$

we get a unique solution for $\lambda(\varphi)$ through integration of (10):

$$\begin{aligned} \lambda(\varphi) &= \tan \beta \cdot \int_{\varphi_0}^{\varphi} \frac{1}{\cos t} dt + \lambda_0 = \tan \beta \cdot \left(\int_0^{\varphi} \frac{1}{\cos t} dt - \int_0^{\varphi_0} \frac{1}{\cos t} dt \right) + \lambda_0 \\ &= (\text{arggd } \varphi - \text{arggd } \varphi_0) \cdot \tan \beta + \lambda_0. \end{aligned}$$

Here, arggd is the inverse Gudermann function given by (14). This representation does not include the special case $\beta = \pm \frac{\pi}{2}$, where the loxodrome is a circle of latitude which cannot be parameterized by latitude (which is constant).

Finally, $\vec{\gamma} := \vec{\tilde{\gamma}} \circ \tilde{\varphi}^{-1}$ is a representation for a loxodrome parameterized by latitude:

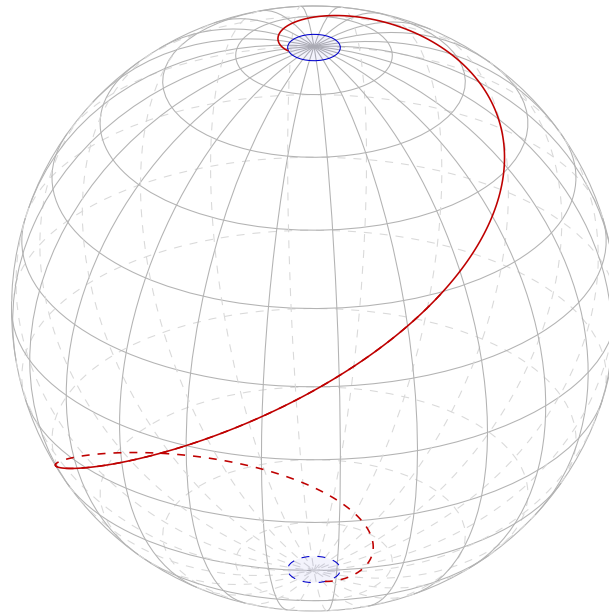


Figure 3: Loxodrome with a constant angle of 60° to the north. Also, circles of latitude at 85.0511° N and 85.0511° S are illustrated in blue. These are boundaries for the Web Mercator projection described in Section 3.

Loxodrome representation by latitude

A loxodrome with a constant angle $\beta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ to the north, running from the south pole to the north pole through $\vec{r}(\varphi_0, \lambda_0)$ with $\lambda_0 \in \mathbb{R}$ and $\varphi_0 \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, which is parameterized by latitude, has a representation

$$\vec{\gamma} :]-\frac{\pi}{2}, \frac{\pi}{2}[\rightarrow \mathbb{R}^3, \quad \varphi \mapsto \vec{\gamma}(\varphi) = \vec{r}(\varphi, \lambda(\varphi)),$$

where

$$\lambda(\varphi) = (\operatorname{arggd} \varphi - \operatorname{arggd} \varphi_0) \cdot \tan \beta + \lambda_0. \tag{11}$$

arggd is the inverse Gudermann function given by (14).

An example illustration for such a loxodrome is given by Figure 3.

Alternatively, when (11) is dissolved for φ , parametrization can be changed to longitude:

Loxodrome representation by longitude

A loxodrome with a constant angle $\beta \in]0, \pi[$ to the north, running through $\vec{r}(\varphi_0, \lambda_0)$ with $\lambda_0 \in \mathbb{R}$ and $\varphi_0 \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, which is parameterized by longitude, has a representation

$$\vec{\gamma}^* : \mathbb{R} \rightarrow \mathbb{R}^3, \quad \lambda \mapsto \vec{\gamma}^*(\lambda) = \vec{r}(\varphi(\lambda), \lambda),$$

where

$$\varphi(\lambda) = \operatorname{gd}((\lambda - \lambda_0) \cdot \cot \beta + \operatorname{arggd} \varphi_0). \tag{12}$$

gd is the Gudermann function given by (13) and arggd is the inverse Gudermann function given by (14).

In this representation with $\beta \in]0, \pi[$ the parameterized curve $\vec{\gamma}^*$ has a heading to the east.

2.2 Excursion: Gudermann Function and inverse Gudermann Function

As a short excursion, we consider the Gudermann or Gudermannian function gd (Weisstein 2020) also known as Mercator function (Walz 2017) given by (13) and its inverse arggd (which we will see) by (14). The following tabular collocation lists these functions with several equivalent expressions.

Gudermann function gd	Inverse Gudermann function arggd
$\text{gd} : \mathbb{R} \rightarrow \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[,$ $x \mapsto \text{gd } x = \int_0^x \frac{1}{\cosh t} dt \quad (13)$	$\text{arggd} : \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[\rightarrow \mathbb{R},$ $x \mapsto \text{arggd } x = \int_0^x \frac{1}{\cos t} dt \quad (14)$
$\frac{d}{dx} \text{gd } x = \frac{1}{\cosh x} \quad (15)$	$\frac{d}{dx} \text{arggd } x = \frac{1}{\cos x} \quad (16)$
$\text{gd } x = \arctan(\sinh x) \quad (17)$	$\text{arggd } x = \text{arsinh}(\tan x) \quad (18)$
	$\text{arggd } x = \ln\left(\tan x + \frac{1}{\cos x}\right) \quad (19)$
$\text{gd } x = \arcsin\left(\frac{e^{2x} - 1}{e^{2x} + 1}\right) \quad (20)$	$\text{arggd } x = \frac{1}{2} \ln\left(\frac{1 + \sin x}{1 - \sin x}\right) \quad (21)$
$\text{gd } x = \arcsin(\tanh x) \quad (22)$	$\text{arggd } x = \text{artanh}(\sin x) \quad (23)$
$\text{gd } x = 2 \arctan\left(\frac{e^x - 1}{e^x + 1}\right) \quad (24)$	$\text{arggd } x = \ln\left(\frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}}\right) \quad (25)$
$\text{gd } x = 2 \arctan\left(\tanh \frac{x}{2}\right) \quad (26)$	$\text{arggd } x = 2 \text{artanh}\left(\tan \frac{x}{2}\right) \quad (27)$
$\text{gd } x = 2 \arctan(e^x) - \frac{\pi}{2} \quad (28)$	$\text{arggd } x = \ln\left(\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right) \quad (29)$

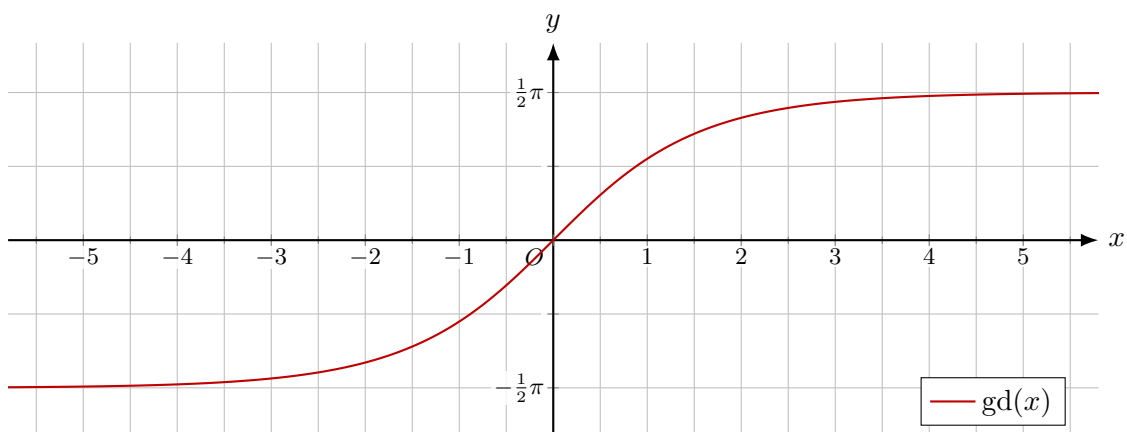


Figure 4: Graph of the Gudermann function gd .

Starting with definition (13), we proof all other expressions by elementary manipulations.

$$\begin{aligned} \frac{d}{dx} \operatorname{gd} x &= \frac{d}{dx} \int_0^x \frac{1}{\cosh t} dt = \frac{1}{\cosh x}. && \text{This gives (15).} \\ \operatorname{gd} x &= \int_0^x \frac{1}{\cosh t} dt = \int_0^x \frac{\cosh t}{\cosh^2 t} dt = \int_0^x \frac{\cosh t}{1 + \sinh^2 t} dt = [\arctan(\sinh t)]_0^x \\ &= \arctan(\sinh x). && \text{This gives (17).} \end{aligned}$$

gd is a strictly monotonic increasing function, because $\frac{d}{dx} \operatorname{gd} x = \frac{1}{\cosh x} > 0$ for all $x \in \mathbb{R}$. Together with

$$\left. \begin{aligned} \lim_{x \rightarrow -\infty} \operatorname{gd} x &= \lim_{x \rightarrow -\infty} \arctan(\sinh x) = -\frac{\pi}{2} \\ \lim_{x \rightarrow \infty} \operatorname{gd} x &= \lim_{x \rightarrow \infty} \arctan(\sinh x) = \frac{\pi}{2} \end{aligned} \right\} \operatorname{gd}(\mathbb{R}) = \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[,$$

we have proven (13) to be well-defined. Also, gd is shown to be bijective. Figure 4 shows the graph of gd .

Now, we compute the inverse by manipulation of (17):

$$y = \operatorname{gd} x = \arctan(\sinh x) \quad \Rightarrow \quad \tan y = \sinh x \quad \Rightarrow \quad x = \operatorname{arsinh}(\tan y).$$

This gives

$$\operatorname{gd}^{-1} : \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[\rightarrow \mathbb{R}, \quad x \mapsto \operatorname{gd}^{-1}(x) = \operatorname{arsinh}(\tan x).$$

The derivative of this inverse is

$$\begin{aligned} \frac{d}{dx} \operatorname{gd}^{-1}(x) &= \frac{d}{dx} (\operatorname{arsinh}(\tan x)) = \frac{1}{\sqrt{\tan^2 x + 1}} \cdot \frac{1}{\cos^2 x} = \frac{1}{\sqrt{\sin^2 x + \cos^2 x}} \cdot \frac{1}{\cos x} \\ &= \frac{1}{\cos x}. \end{aligned}$$

Here, we used that $\cos x > 0$ for $x \in \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[$. Now, we have with (14) that

$$\begin{aligned} \operatorname{arggd} x &= \int_0^x \frac{1}{\cos t} dt = \operatorname{gd}^{-1}(x) + C, \\ \text{and } C &= \operatorname{arggd}(0) - \operatorname{gd}^{-1}(0) = 0 - \operatorname{arsinh}(\tan 0) = 0. \end{aligned}$$

Altogether, this shows $\operatorname{arggd} = \operatorname{gd}^{-1}$ (14). Also, we have (16) and (18).

We remember that $\cos x > 0$ for $x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ and we continue with transforming (18):

$$\begin{aligned} \operatorname{arggd} x &= \operatorname{arsinh}(\tan x) = \ln\left(\tan x + \sqrt{\tan^2 x + 1}\right) = \ln\left(\tan x + \sqrt{\frac{1}{\cos^2 x}}\right) \\ &= \ln\left(\tan x + \frac{1}{\cos x}\right) && \text{This gives (19)} \\ &= \ln\left(\frac{\sin x + 1}{\cos x}\right) = \frac{1}{2} \ln\left(\frac{(1 + \sin x)^2}{\cos^2 x}\right) = \frac{1}{2} \ln\left(\frac{(1 + \sin x)^2}{1 - \sin^2 x}\right) \\ &= \frac{1}{2} \ln\left(\frac{1 + \sin x}{1 - \sin x}\right) && \text{This gives (21)} \\ &= \operatorname{artanh}(\sin x) && \text{This gives (23)} \\ &= \frac{1}{2} \ln\left(\frac{1 + 2 \sin \frac{x}{2} \cos \frac{x}{2}}{1 - 2 \sin \frac{x}{2} \cos \frac{x}{2}}\right) = \frac{1}{2} \ln\left(\frac{\cos^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2} + \sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2} - 2 \sin \frac{x}{2} \cos \frac{x}{2} + \sin^2 \frac{x}{2}}\right) \\ &= \frac{1}{2} \ln\left(\frac{(\cos \frac{x}{2} + \sin \frac{x}{2})^2}{(\cos \frac{x}{2} - \sin \frac{x}{2})^2}\right) = \ln\left(\frac{\cos \frac{x}{2} + \sin \frac{x}{2}}{\cos \frac{x}{2} - \sin \frac{x}{2}}\right) \\ &= \ln\left(\frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}}\right) && \text{This gives (25)} \\ &= 2 \operatorname{artanh}\left(\tan \frac{x}{2}\right). && \text{This gives (27)} \end{aligned}$$

We have used $\cos \frac{x}{2} > \sin \frac{x}{2}$ for $\frac{x}{2} \in]-\frac{\pi}{4}, \frac{\pi}{4}[$ during the transformation.

Now, we look again at (25):

$$\begin{aligned} \operatorname{arggd} x &= \ln\left(\frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}}\right) = \ln\left(\frac{\tan \frac{\pi}{4} + \tan \frac{x}{2}}{1 - \tan \frac{\pi}{4} \tan \frac{x}{2}}\right) \\ &= \ln\left(\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right). && \text{This gives (29)} \end{aligned}$$

(23) yields directly (22), i.e.

$$\begin{aligned} \operatorname{gd} x &= \operatorname{arcsin}(\tanh x) = \operatorname{arcsin}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) \\ &= \operatorname{arcsin}\left(\frac{e^{2x} - 1}{e^{2x} + 1}\right). && \text{This gives (20)} \end{aligned}$$

(29) yields directly (28) and (27) yields directly (26), i.e.

$$\begin{aligned} \operatorname{gd} x &= 2 \arctan\left(\tanh \frac{x}{2}\right) = 2 \arctan\left(\frac{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}{e^{\frac{x}{2}} + e^{-\frac{x}{2}}}\right) \\ &= 2 \arctan\left(\frac{e^x - 1}{e^x + 1}\right). && \text{This gives (24)} \end{aligned}$$

2.3 Mercator Projection of Latitude/Longitude to Cartesian Coordinates

The classic description of a Mercator projection is to project a sphere to a cylinder tangential to it at the equator. Later, this cylinder is rolled flat to a plane (the map). On this map, a part of a loxodrome shall appear as straight line. The following mapping (30) describes this projection,

but its domain of definition is deliberately larger than needed to allow arbitrary longitude values (with a period of 2π).

Mercator projection

Let $s > 0$ be some scaling factor and let $x_{\text{off}}, y_{\text{off}} \in \mathbb{R}$ be some offset values.

The Mercator projection of a pair (φ, λ) consisting of a latitude angle φ and a longitude angle λ to a pair (x, y) of Cartesian coordinates (on a map) is given by

$$M : \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[\times \mathbb{R} \rightarrow \mathbb{R}^2, \quad \begin{pmatrix} \varphi \\ \lambda \end{pmatrix} \mapsto \begin{pmatrix} x(\lambda) \\ y(\varphi) \end{pmatrix}, \quad (30)$$

where

$$x(\lambda) := s \cdot \lambda + x_{\text{off}}, \quad (31)$$

$$y(\varphi) := s \cdot \operatorname{arggd} \varphi + y_{\text{off}}. \quad (32)$$

We now consider a loxodrome on the sphere with a constant angle $\beta \in \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[$ to the north through $\vec{r}(\varphi_0, \lambda_0)$. This point is projected to a point (x_0, y_0) on the plane by

$$x_0 = s \cdot \lambda_0 + x_{\text{off}},$$

$$y_0 = s \cdot \operatorname{arggd} \varphi_0 + y_{\text{off}}.$$

For the loxodrome parameterized by latitude we have with (11):

$$\lambda = (\operatorname{arggd} \varphi - \operatorname{arggd} \varphi_0) \cdot \tan \beta + \lambda_0.$$

For the points (x, y) of the projected loxodrome, this yields the following relationship:

$$\begin{aligned} x &= s \cdot \lambda + x_{\text{off}} = s \cdot ((\operatorname{arggd} \varphi - \operatorname{arggd} \varphi_0) \cdot \tan \beta + \lambda_0) + x_{\text{off}} \\ &= (s \cdot \operatorname{arggd} \varphi + y_{\text{off}}) \cdot \tan \beta - (s \cdot \operatorname{arggd} \varphi_0 + y_{\text{off}}) \cdot \tan \beta + s \cdot \lambda_0 + x_{\text{off}} \\ &= y \cdot \tan \beta - y_0 \cdot \tan \beta + x_0. \\ \Rightarrow \quad \frac{x - x_0}{y - y_0} &= \tan \beta. \end{aligned} \quad (33)$$

This means that the projected loxodrome is a straight line through (x_0, y_0) which has a constant angle β with the y -axis (north direction).

(31) and (32) can be easily transformed to get an inverse projection from the plane (map) back to the sphere:

Inverse Mercator projection

Let $s > 0$ be some scaling factor and let $x_{\text{off}}, y_{\text{off}} \in \mathbb{R}$ be some offset values.

The inverse Mercator projection of a pair (x, y) of Cartesian coordinates to a pair (φ, λ) consisting of a latitude angle φ and a longitude angle λ is given by

$$M^{-1} : \mathbb{R}^2 \rightarrow \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[\times \mathbb{R}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \varphi(y) \\ \lambda(x) \end{pmatrix}, \quad (34)$$

where

$$\varphi(y) := \operatorname{gd} \left(\frac{y - y_{\text{off}}}{s} \right), \quad (35)$$

$$\lambda(x) := \frac{x - x_{\text{off}}}{s}. \quad (36)$$

3 Projection to Map Tiles

The general idea of map tiles is that the planar map is cut into square pieces which themselves are cut further into square pieces. Every cut gives a higher *zoom level* with a growing number of smaller pieces which are called map tiles in the following.

On application side, for every map tile a fixed size pixel graphic file (typically 256 times 256 pixel) is created to be served over the Web. The depicted map is composed of several map tiles from a tile server. The map scale is increased (*zoomed in*) by selecting map tiles from a higher zoom level.

The map tile model uses a simplified reference geoid which is a perfect sphere. This gives a slight error in projecting the ellipsoidal shape of the earth. The size of the error is discussed later in Section 4.

The Mercator projection (30) has \mathbb{R}^2 as image. In the following, the domain of the projection and same scaling are adapted to get $[0, 1]^2$ as image of the projection. This unit square is identical to the only tile of zoom level 0, the root tile for the disassembling into smaller tiles.

3.1 Transformation of Latitude/Longitude to unified Cartesian Coordinates

For the Mercator projection (30), we now consider a scaling factor $s = \frac{1}{2\pi}$ and offset values $x_{\text{off}} = \frac{1}{2}$ and $y_{\text{off}} = \frac{1}{2}$. Also, we clip the longitude and latitude domain such that the projected Cartesian coordinates lie in the unit square $[0, 1]^2$. This results in the following unified projection:

Unified clipped Mercator projection

The unified clipped Mercator projection of a pair (φ, λ) consisting of a latitude angle φ and a longitude angle λ to a pair (x, y) of Cartesian coordinates is given by

$$M_U : [-\text{gd } \pi, \text{gd } \pi] \times [-\pi, \pi] \rightarrow [0, 1]^2, \quad \begin{pmatrix} \varphi \\ \lambda \end{pmatrix} \mapsto \begin{pmatrix} x(\lambda) \\ y(\varphi) \end{pmatrix}, \quad (37)$$

where

$$x(\lambda) := \frac{\lambda}{2\pi} + \frac{1}{2}, \quad (38)$$

$$y(\varphi) := \frac{\text{arggd } \varphi}{2\pi} + \frac{1}{2}. \quad (39)$$

In the same manner, (34) gives the inverse mapping:

Inverse unified clipped Mercator projection

The inverse unified clipped Mercator projection of a pair (x, y) of coordinates to a pair (φ, λ) consisting of a latitude angle φ and a longitude angle λ is given by

$$M_U^{-1} : [0, 1]^2 \rightarrow [-\text{gd } \pi, \text{gd } \pi] \times [-\pi, \pi], \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \varphi(y) \\ \lambda(x) \end{pmatrix} \quad (40)$$

where

$$\varphi(y) := \text{gd}(2\pi y - \pi), \quad (41)$$

$$\lambda(x) := 2\pi x - \pi. \quad (42)$$

The following should be remarked:

- The resulting unit square $[0, 1]^2$ is considered to be the complete presentable world map. On zoom level 0, this is the one and only tile of the map (root tile).
- Parts of the polar regions including the poles are not mapped, because the latitude angle φ is restricted by

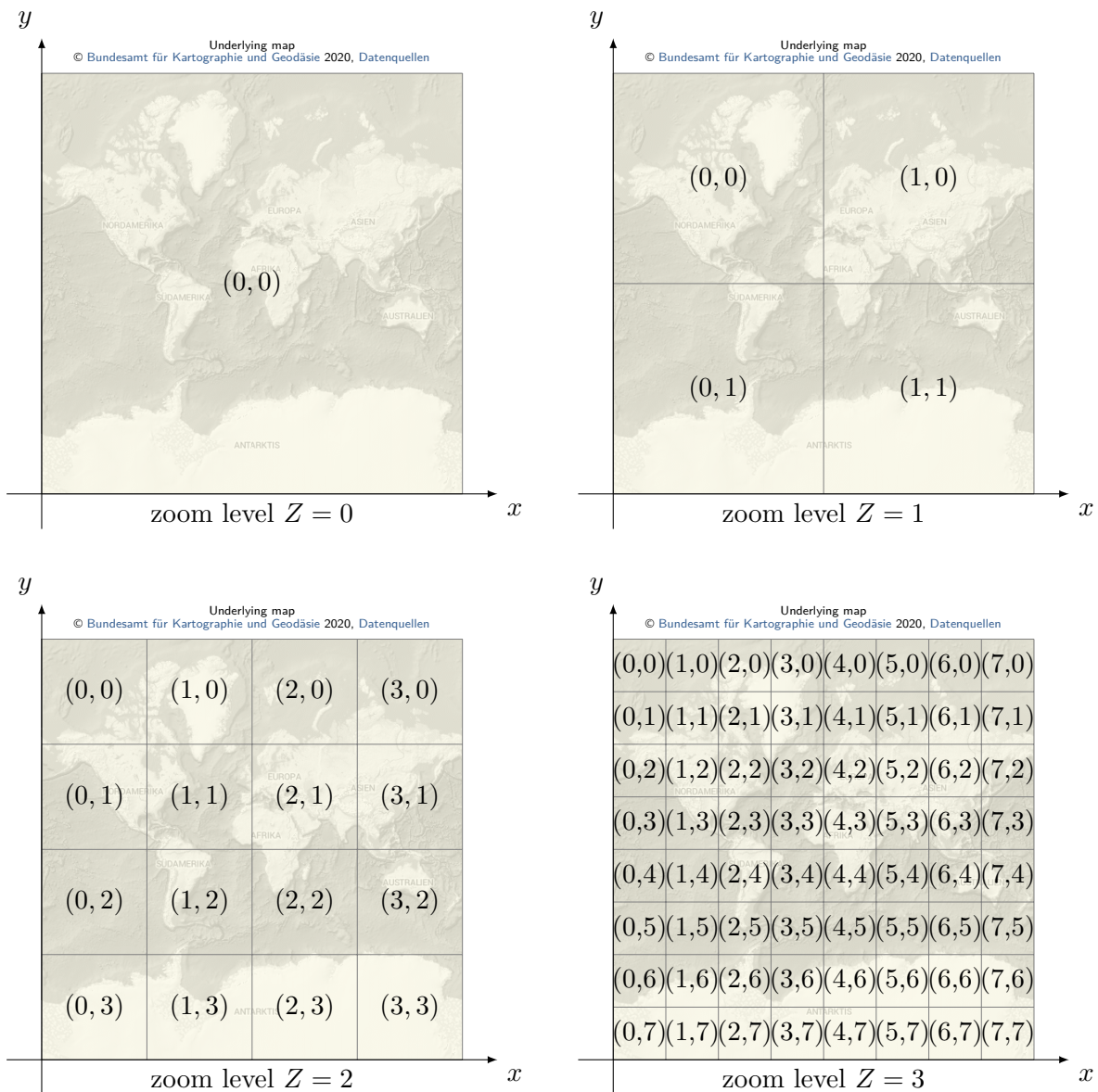
$$\text{gd } \pi \doteq 1.48442 \doteq 85.0511^\circ,$$

i.e. only positions between 85.0511° N and 85.0511° S can be displayed. Figure 3 illustrates these boundaries.

- The longitude angle λ is restricted by π , but if needed, we allow it to be unrestricted. Here, λ has a period of 2π and x has a period of 1.
- Note that $x(\lambda)$ defined by (38) is oriented from south to north, while OpenStreetMap typically uses another representation which conforms to $1 - x(\lambda)$ and which is oriented from north to south (OpenStreetMap contributors 2019).

3.2 Map Tiles on different Zoom Levels

On every zoom level $Z \in \mathbb{N}_0$, the unit square $[0, 1]^2$ map is disassembled into 2^Z times 2^Z quadratic tiles. Each tile on zoom level Z is addressed by a coordinate pair (X, Y) with $X, Y \in \{0, \dots, 2^Z - 1\}$ according to the following scheme:



Every map tile $T(Z, X, Y)$ is a subset of $[0, 1]^2$. According to the scheme above, the following definition (43) for $T(Z, X, Y)$ can be derived.

Map tile $T(Z, X, Y)$

The triple (Z, X, Y) denotes a map tile $T(Z, X, Y)$ with zoom level $Z \in \mathbb{N}_0$ and coordinate pair (X, Y) . With $N := 2^Z$, for every $X, Y \in \{0, N - 1\}$ define a map tile

$$\begin{aligned}
 T(Z, X, Y) &:= \left[\frac{X}{N}, \frac{X+1}{N} \right] \times \left[\frac{N-Y-1}{N}, \frac{N-Y}{N} \right] \\
 &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid \frac{X}{N} \leq x \leq \frac{X+1}{N}, \frac{N-Y-1}{N} \leq y \leq \frac{N-Y}{N} \right\}
 \end{aligned}
 \tag{43}$$

as subset of $[0, 1]^2$.

As an extension, we set

$$T(Z, X + kN, Y) := T(Z, X, Y) \quad \text{for all } k \in \mathbb{Z}.$$

Using (40), (41), and (42), the latitude and longitude for the tile borders can be computed, resulting in the following:

Latitude range and longitude range of map tile $T(Z, X, Y)$

Let zoom level $Z \in \mathbb{N}_0$, $N := 2^Z$, and let $X, Y \in \{0, N - 1\}$. Then,

$$M_U^{-1}(T(Z, X, Y)) = [\varphi_{\min}, \varphi_{\max}] \times [\lambda_{\min}, \lambda_{\max}]$$

contains all positions on the sphere relating to map tile $T(Z, X, Y)$, where

$$\begin{aligned}
 \varphi_{\min} &= \text{gd} \left(2\pi \frac{N-Y-1}{N} - \pi \right) = \text{gd} \left(\pi \frac{N-2Y-2}{N} \right), \\
 \varphi_{\max} &= \text{gd} \left(2\pi \frac{N-Y}{N} - \pi \right) = \text{gd} \left(\pi \frac{N-2Y}{N} \right), \\
 \lambda_{\min} &= 2\pi \frac{X}{N} - \pi = \pi \frac{2X-N}{N}, \\
 \lambda_{\max} &= 2\pi \frac{X+1}{N} - \pi = \pi \frac{2X+2-N}{N}.
 \end{aligned}$$

4 Deviation from the WGS 84 Ellipsoid and Size Considerations

For Web Mercator, the ellipsoidal coordinates for WGS 84 are developed on a sphere. As seen before, the projection to map tiles does not require any radius value for the sphere. A concrete radius is needed when the map is populated with scale values, distances, map height and width, etc.

EPSG Geodetic Parameter Registry 2020 states “Relative to WGS 84 / World Mercator (CRS code 3395) errors of 0.7 percent in scale and differences in northing of up to 43km in the map (equivalent to 21km on the ground) may arise.” This statement refers to the convention that the value of the Web Mercator radius is equal to the semi-major axis of the WGS 84 ellipsoid. This radius is also used by OpenStreetMap contributors 2019. But, according to Battersby, Finn, Usery, and Yamamoto 2014 it is not clear, if all applications and services use this radius. One example is www.netzwolf.info 2009 where presumably the volumetric radius of the Earth is used. Besides the volumetric radius (equal volume), other candidates are the authalic radius (equal area), and the mean radius $\frac{2a+b}{3}$ (Battersby, Finn, Usery, and Yamamoto 2014).

The WGS 84 Ellipsoid (Spatialreference.org 2020a; Spatialreference.org 2020b; National Geospatial-Intelligence Agency 2014) modelling the Earth is defined by

$$\text{ellipsoid semi-major axis } a = 6\,378\,137 \text{ m}, \quad (44)$$

$$\text{ellipsoid inverse flattening } \frac{1}{f} = 298.257\,223\,563, \quad (45)$$

$$\text{ellipsoid eccentricity } e = \sqrt{2f - f^2} = 0.081\,819\,190\,842\,6, \quad (46)$$

$$e^2 = 2f - f^2 = 0.006\,694\,379\,990\,14, \quad (47)$$

$$1 - e^2 = 2f - f^2 = 0.999\,988\,758\,661, \quad (48)$$

$$\text{ellipsoid semi-minor axis } b = a(1 - f) = 6\,356\,752.314\,252 \text{ m}. \quad (49)$$

Also, the Web Mercator sphere has a radius equal to the ellipsoid semi-major axis, i.e.

$$\text{sphere radius } R_0 = 6\,378\,137 \text{ m}. \quad (50)$$

Further, following the naming of Moritz 2000, we note for future discussions the

$$\text{mean radius } R_1 = \frac{2a + b}{3} = 6\,371\,008.7714 \text{ m}, \quad (51)$$

$$\text{authalic average radius } R_2 = 6\,371\,007.1810 \text{ m}, \quad (52)$$

$$\text{volumetric average radius } R_3 = \sqrt[3]{a^2b} = 6\,371\,000.7900 \text{ m}. \quad (53)$$

In the following, a short overview construction of ellipsoidal coordinates is given. Then, deviations of the Web Mercator sphere to the WGS 84 ellipsoid are discussed.

4.1 Ellipsoidal Coordinates

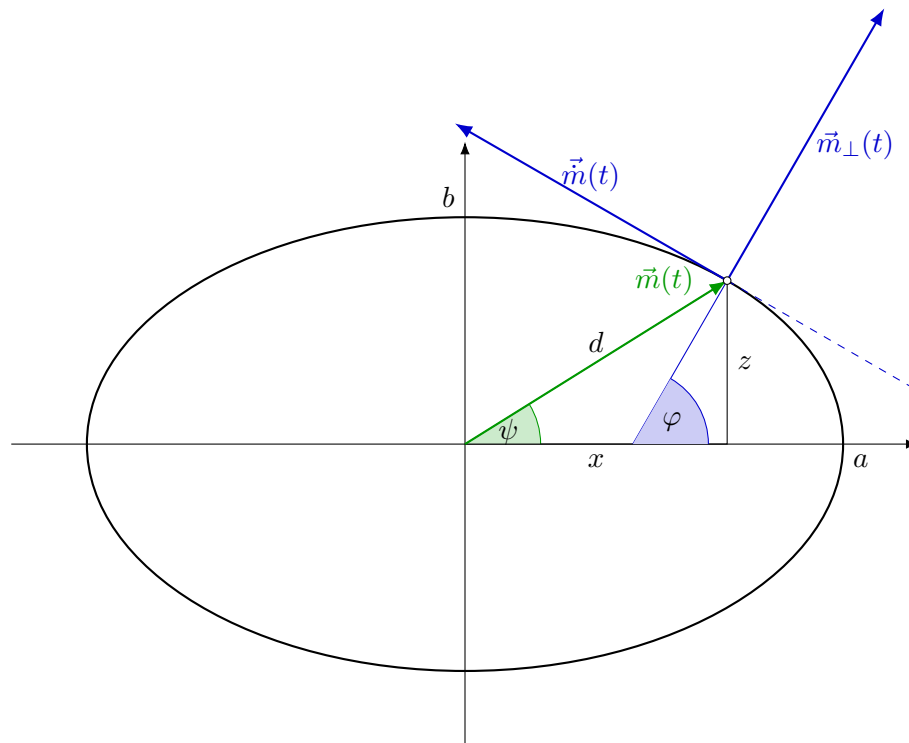


Figure 5: Ellipsoidal coordinates. Ellipse as intersection of an ellipsoid of rotation with the x - z -plane.

Since the WGS 84 ellipsoid is an ellipsoid of rotation, it is enough to regard an intersection with the x - z -plane which gives an ellipse, see Figure 5.

Here, φ is the geodetic latitude while ψ is a spherical polar angle (geocentric latitude) with angle-dependent geocentric radius d (Heck and Seitz 2017). x and z are the Cartesian coordinates of the considered point on the meridian ellipse. x is also the radius of the circle of latitude φ for the WGS 84 ellipsoid.

We start with an auxiliary parametrization of the meridian ellipse given by

$$\vec{m}(t) = \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} a \cos t \\ b \sin t \end{pmatrix}, \quad t \in \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[\tag{54}$$

which obviously fulfills $\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1$. This parameterizes the right-hand half of our ellipse by parametric latitude t .

Here, we have

$$\tan \psi = \frac{z}{x} = \frac{b}{a} \tan t. \tag{55}$$

The tangent vector at $\vec{m}(t)$ is found as

$$\vec{m}(t) = \begin{pmatrix} -a \sin t \\ b \cos t \end{pmatrix}$$

and further the normal vector on the ellipse as

$$\vec{m}_\perp(t) = \begin{pmatrix} b \cos t \\ a \sin t \end{pmatrix}.$$

The geodetic latitude φ is the angle between $\vec{m}_\perp(t)$ and the equatorial plane, i.e.

$$\tan \varphi = \frac{a \sin t}{b \cos t} = \frac{a}{b} \tan t. \quad (56)$$

$$t = \arctan \left(\frac{b}{a} \tan \varphi \right). \quad (57)$$

Combining (55) and (56) gives

$$\tan \psi = \frac{b^2}{a^2} \tan \varphi = (1 - e^2) \tan \varphi \quad (58)$$

using the eccentricity definition

$$e^2 = \frac{a^2 - b^2}{a^2} = 1 - \frac{b^2}{a^2}, \quad \frac{b^2}{a^2} = 1 - e^2, \quad b = a\sqrt{1 - e^2}, \quad a = b \frac{1}{\sqrt{1 - e^2}}.$$

From (58), we get directly

$$\psi = \arctan \left((1 - e^2) \tan \varphi \right). \quad (59)$$

Let us note the basic relations (for every $\xi \in \mathbb{R}$)

$$\sin^2(\arctan \xi) = \frac{\xi^2}{1 + \xi^2} \quad \text{and} \quad \cos^2(\arctan \xi) = \frac{1}{1 + \xi^2} \quad (60)$$

to deduct from (54) and (57) the following

$$\begin{aligned} x^2 &= a^2 \cos^2 t = a^2 \cos^2 \left(\arctan \left(\frac{b}{a} \tan \varphi \right) \right) \\ &= a^2 \frac{1}{1 + \frac{b^2}{a^2} \tan^2 \varphi} = \frac{a^2 \cos^2 \varphi}{\cos^2 \varphi + \frac{b^2}{a^2} \sin^2 \varphi} \\ &= \frac{a^2 \cos^2 \varphi}{1 - \sin^2 \varphi + \frac{b^2}{a^2} \sin^2 \varphi} = \frac{a^2 \cos^2 \varphi}{1 - e^2 \sin^2 \varphi}. \end{aligned}$$

Since x and $\cos \varphi$ are both positive values, this gives

$$x = \frac{a \cos \varphi}{\sqrt{1 - e^2 \sin^2 \varphi}}. \quad (61)$$

z is computed in the same manner from (54) and (57) by

$$\begin{aligned} z^2 &= b^2 \sin^2 t = b^2 \sin^2 \left(\arctan \left(\frac{b}{a} \tan \varphi \right) \right) \\ &= b^2 \frac{\frac{b^2}{a^2} \tan^2 \varphi}{1 + \frac{b^2}{a^2} \tan^2 \varphi} = \frac{\frac{b^4}{a^2} \sin^2 \varphi}{\cos^2 \varphi + \frac{b^2}{a^2} \sin^2 \varphi} = \frac{a^2(1 - e^2)^2 \sin^2 \varphi}{1 - e^2 \sin^2 \varphi}. \end{aligned}$$

Since z and $\sin \varphi$ share the same algebraic sign, this gives

$$z = \frac{a(1 - e^2) \sin \varphi}{\sqrt{1 - e^2 \sin^2 \varphi}}. \quad (62)$$

For sake of completeness, d is found as length of $\vec{m}(t)$ from (54), (61), and (62) by

$$\begin{aligned} d &= \sqrt{x^2 + z^2} = \frac{a}{\sqrt{1 - e^2 \sin^2 \varphi}} \sqrt{\cos^2 \varphi + (1 - e^2)^2 \sin^2 \varphi} \\ &= \frac{a}{\sqrt{1 - e^2 \sin^2 \varphi}} \sqrt{1 - \sin^2 \varphi + (1 - 2e^2 + e^4) \sin^2 \varphi} \\ &= \frac{a}{\sqrt{1 - e^2 \sin^2 \varphi}} \sqrt{1 - (2 - e^2)e^2 \sin^2 \varphi}. \end{aligned} \tag{63}$$

For the summary, the ellipse is to be rotated around the z -axis to get the ellipsoid. From (61) and (62) we get by adding trigonometric functions of the longitude:

Position vector for the ellipsoid

The position vector \vec{r} for the ellipsoid points can be given in dependency of a latitude angle $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and a longitude angle $\lambda \in \mathbb{R}$ as

$$\vec{r}(\varphi, \lambda) = \begin{pmatrix} A(\varphi) \cdot \cos \lambda \cdot \cos \varphi \\ A(\varphi) \cdot \sin \lambda \cdot \cos \varphi \\ A(\varphi) \cdot (1 - e^2) \cdot \sin \varphi \end{pmatrix} \tag{64}$$

where

$$A(\varphi) := \frac{a}{\sqrt{1 - e^2 \sin^2 \varphi}}. \tag{65}$$

We consider now the signed meridian distance $\mu(\varphi)$ from latitude φ to the equator following a meridian. With (54) and (57), it is found as

$$\mu(\varphi) = \int_0^{t(\varphi)} \|\vec{m}(t)\| dt = \int_0^{t(\varphi)} \left(a^2 \sin^2 t + b^2 \cos^2 t \right)^{\frac{1}{2}} dt, \quad \text{with } t(\varphi) = \arctan \left(\frac{b}{a} \tan \varphi \right). \tag{66}$$

From (62) we get

$$a^2 \sin^2 t = \frac{a^2}{b^2} z^2 = \frac{a^2 a^2 (1 - e^2)^2 \sin^2 \varphi}{b^2 (1 - e^2 \sin^2 \varphi)} = \frac{b^2 \sin^2 \varphi}{1 - e^2 \sin^2 \varphi} \tag{67}$$

and from (61) we get

$$b^2 \cos^2 t = \frac{b^2}{a^2} x^2 = \frac{b^2 a^2 \cos^2 \varphi}{a^2 (1 - e^2 \sin^2 \varphi)} = \frac{b^2 \cos^2 \varphi}{1 - e^2 \sin^2 \varphi}. \tag{68}$$

Differentiating (57) gives

$$\frac{dt}{d\varphi}(\varphi) = \frac{1}{1 + \frac{b^2}{a^2} \tan^2 \varphi} \cdot \frac{b}{a} \cdot \frac{1}{\cos^2 \varphi} = \frac{b}{a} \cdot \frac{1}{\cos^2 \varphi + \frac{b^2}{a^2} \sin^2 \varphi} = \frac{b}{a} \cdot \frac{1}{1 - e^2 \sin^2 \varphi} \tag{69}$$

Using (67), (68), and (69) for integral substitution in (66) results in

$$\begin{aligned} \mu(\varphi) &= \int_0^{\varphi} \left(\frac{b^2 \cos^2 \phi}{1 - e^2 \sin^2 \phi} + \frac{b^2 \cos^2 \phi}{1 - e^2 \sin^2 \phi} \right)^{\frac{1}{2}} \cdot \frac{b}{a} \cdot \frac{1}{1 - e^2 \sin^2 \phi} d\phi \\ &= \frac{b^2}{a} \int_0^{\varphi} \frac{1}{(1 - e^2 \sin^2 \phi)^{\frac{3}{2}}} d\phi = a(1 - e^2) \int_0^{\varphi} \frac{d\phi}{(1 - e^2 \sin^2 \phi)^{\frac{3}{2}}}. \end{aligned}$$

Besides numerical integration for this elliptic integral, there are several known series expansions to evaluate $\mu(\varphi)$, see Osborne 2013, Section 5.8.

Distances on the ellipsoid

The signed parallel distance on the ellipsoid from longitude λ_1 to λ_2 on a circle of latitude φ is given by

$$s_{\text{parallel}} = \frac{a \cos \varphi}{\sqrt{1 - e^2 \sin^2 \varphi}} \cdot (\lambda_2 - \lambda_1) = A(\varphi) \cdot \cos \varphi \cdot (\lambda_2 - \lambda_1). \quad (70)$$

The signed meridian distance on the ellipsoid from latitude φ_1 to φ_2 is given by

$$s_{\text{meridian}} = a(1 - e^2) \int_{\varphi_1}^{\varphi_2} \frac{d\phi}{(1 - e^2 \sin^2 \phi)^{\frac{3}{2}}} = \mu(\varphi_2) - \mu(\varphi_1) \quad (71)$$

where

$$\mu(\varphi) = a(1 - e^2) \int_0^{\varphi} \frac{d\phi}{(1 - e^2 \sin^2 \phi)^{\frac{3}{2}}}. \quad (72)$$

The development of (72) was done immediately beforehand. (70) is a direct consequence from the radius of a circle of latitude φ computed in (61) or derived from (64).

4.2 Deviation of Web Mercator from the WGS 84 Ellipsoid

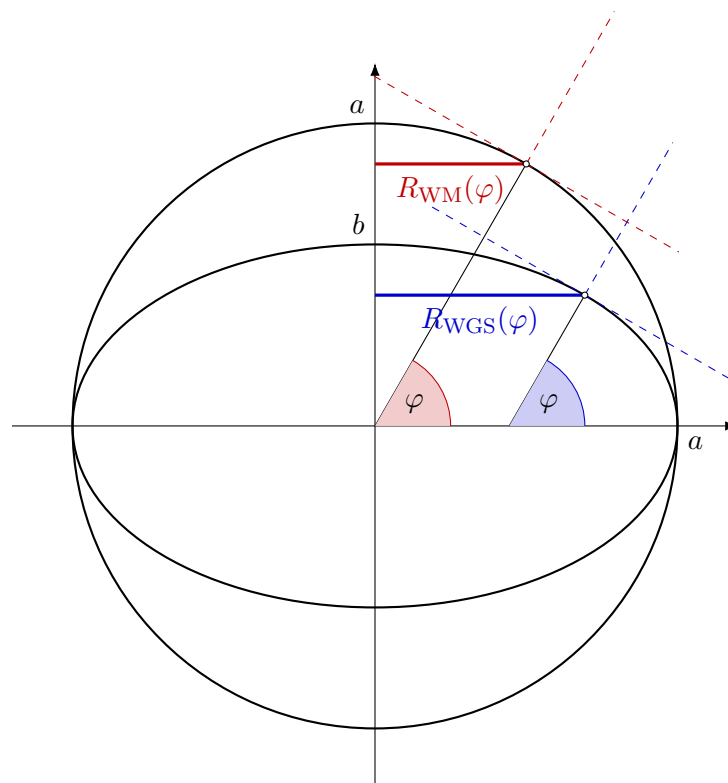


Figure 6: Intersection of the x - z -plane with an exaggerative flattened ellipsoid representing WGS 84 and a Web Mercator sphere with radius a . The displayed points on ellipse and sphere share the same geodetic latitude and are identified with each other.

For Web Mercator, positions on the WGS 84 ellipsoid are interpreted as positions on a sphere, see Figure 6. This introduces deviations (friendly for errors) in map presentation and calculations of distances and scales.

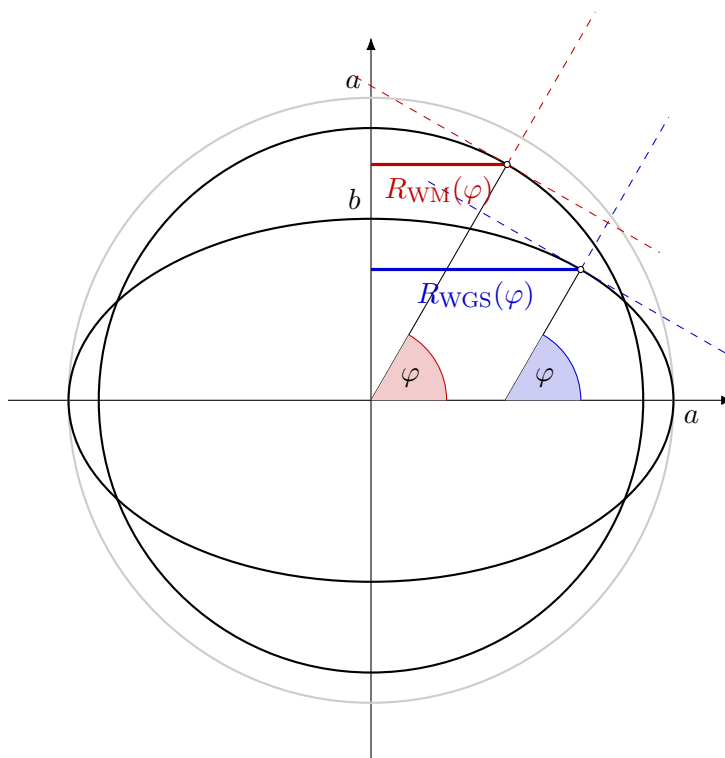


Figure 7: Intersection of the x - z -plane with an exaggerative flattened ellipsoid representing WGS 84 and a Web Mercator sphere with radius small than a . The displayed points on ellipse and sphere share the same geodetic latitude and are identified with each other.

We consider the following distances on the sphere as subject for analysis.

Distances on the sphere	
The signed parallel distance on the sphere with radius R from longitude λ_1 to λ_2 on a circle of latitude φ is given by	
$\hat{s}_{\text{parallel}} = R \cdot \cos \varphi \cdot (\lambda_2 - \lambda_1).$	(73)
The signed meridian distance on the sphere from latitude φ_1 to φ_2 is given by	
$\hat{s}_{\text{meridian}} = R \cdot (\varphi_2 - \varphi_1).$	(74)

Figure 6 displays a position with latitude φ on an ellipsoid representing WGS 84, but very exaggerated. The same latitude φ is seen on a sphere representing Web Mercator which is identified with WGS 84. The radii of circles of latitude are denoted $R_{\text{WGS}}(\varphi)$ and $R_{\text{WM}}(\varphi)$ here.

The pictogram of Figure 6 shows directly that the parallel distance on the sphere is smaller than the parallel distance on the ellipsoid, because $R_{\text{WM}}(\varphi) < R_{\text{WGS}}(\varphi)$. On the other hand, the meridian distance (at least to the equator) on the sphere is larger than the parallel distance on the ellipsoid.

Depending on what effect is seen worse, one could choose a smaller radius for the Web Mercator sphere than the semi-major axis of the ellipsoid. This situation is demonstrated in Figure 7. In comparison to Figure 6, the parallel distance on the sphere is made smaller again, i.e. more deviating from the ellipsoid. Also the meridian distance of the sphere is downsized which *could* decrease the gap to the ellipsoid.

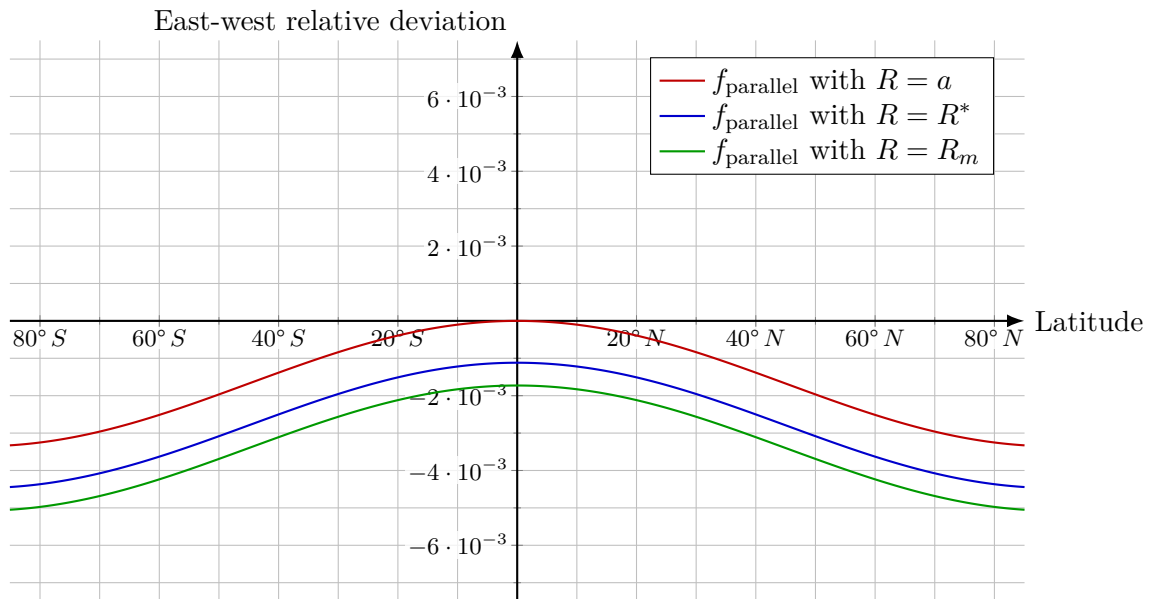


Figure 8: Relative deviation of parallel distances for the same latitude between Web Mercator spheres with different radius to the WGS 84 ellipsoid.

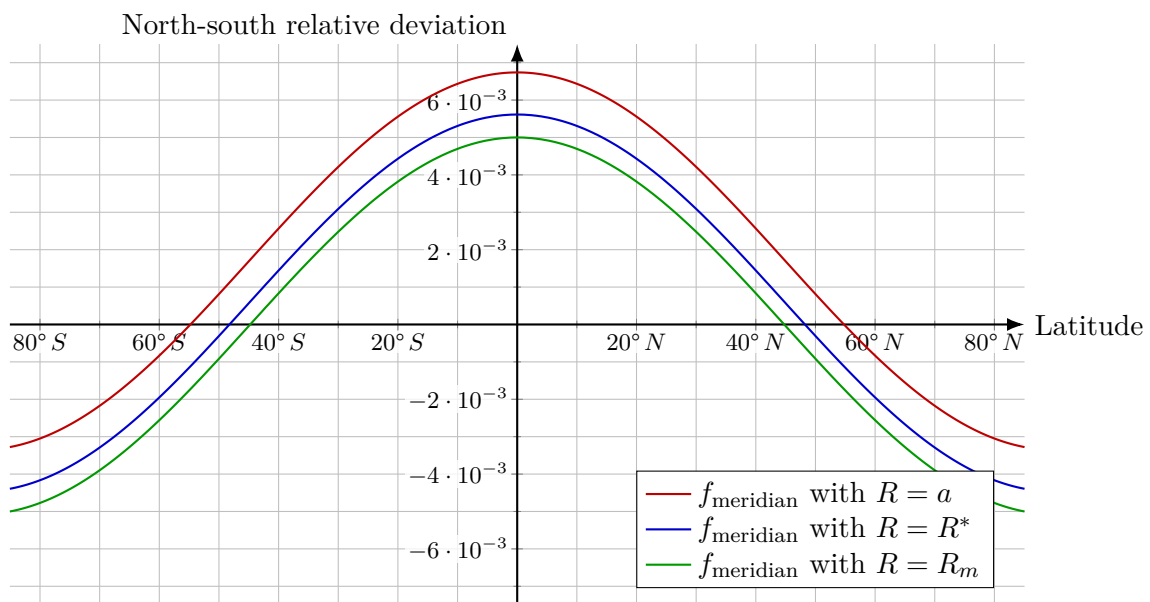


Figure 9: Relative deviation of meridian distances between Web Mercator spheres with different radius to the WGS 84 ellipsoid.

The three “natural” candidates for a uniform radius are the mean radius R_1 (51), the authalic average radius R_2 (52), and the volumetric average radius R_3 (53). All three are very similar with identical first five significant digits. Also, as will be seen, a distinct smaller radius could also be a candidate. Therefore, we define a further

$$\text{average radius } R^* := 6\,371\,000 \text{ m}, \tag{75}$$

which equals to R_1, R_2, R_3 with five to six significant digits.

Firstly, we consider the relative deviation of arcs on a circle of latitude φ from longitude λ_1 to

λ_2 (east-west). With (70) and (73) it is given by

$$\begin{aligned}
 f_{\text{parallel}} &= \frac{\hat{s}_{\text{parallel}}}{s_{\text{parallel}}} - 1 = \frac{R \cdot \cos \varphi \cdot (\lambda_2 - \lambda_1)}{A(\varphi) \cdot \cos \varphi \cdot (\lambda_2 - \lambda_1)} - 1 \\
 &= \frac{R}{a} \cdot \sqrt{1 - e^2 \sin^2 \varphi} - 1.
 \end{aligned}
 \tag{76}$$

This function in dependence of latitude is the relative ratio of the radii of sphere and ellipsoid seen in Figure 6 and Figure 7. For different sphere radii, f_{parallel} is depicted in Figure 8 where the highest absolute deviation is found in the high north and south at $\text{gd } \pi$ and $\text{gd } -\pi$.

Secondly, we consider the average relative deviation of arcs on meridians from latitude φ_1 to φ_2 (north-south). With (71) and (74) it is given by

$$\bar{f}_{\text{meridan}} = \frac{\hat{s}_{\text{meridian}}}{s_{\text{meridian}}} - 1 = \frac{R \cdot (\varphi_2 - \varphi_1)}{\mu(\varphi_2) - \mu(\varphi_1)} - 1.$$

The local relative deviation for a latitude φ is found as limit result

$$\begin{aligned}
 f_{\text{meridan}} &= \lim_{\Delta\varphi \rightarrow 0} \frac{R \cdot (\varphi + \Delta\varphi - \varphi)}{\mu(\varphi + \Delta\varphi) - \mu(\varphi)} - 1 = R \cdot \frac{1}{\lim_{\Delta\varphi \rightarrow 0} \frac{\mu(\varphi + \Delta\varphi) - \mu(\varphi)}{\Delta\varphi}} - 1 \\
 &= R \cdot \frac{1}{\frac{d\mu(\varphi)}{d\varphi}} - 1 = R \cdot \frac{1}{a(1 - e^2) (1 - e^2 \sin^2 \varphi)^{-\frac{3}{2}}} - 1 \\
 &= \frac{R}{a} \cdot \frac{(1 - e^2 \sin^2 \varphi)^{\frac{3}{2}}}{1 - e^2} - 1
 \end{aligned}
 \tag{77}$$

$$= (f_{\text{parallel}} + 1) \cdot \frac{1 - e^2 \sin^2 \varphi}{1 - e^2} - 1.
 \tag{78}$$

This function in dependence of latitude is depicted in Figure 9 where the highest positive deviation is found at the equator and the lowest negative deviation in the high north and south at $\text{gd } \pi$ and $\text{gd } -\pi$.

The results of the analysis are

- If $R = a$ (semi-major axis) is chosen as radius for the Web Mercator sphere, east-west calculations with spherical formulas give an error smaller than

$$\left| \sqrt{1 - e^2 \sin^2(\text{gd } \pi)} - 1 \right| = \left| \sqrt{1 - e^2 \tanh^2 \pi} - 1 \right| \doteq 0.333\%$$

North-south calculations give an error smaller than

$$\left| \frac{1}{1 - e^2} - 1 \right| \doteq 0.674\%$$

This error is smaller than 0.5% north of 25° N and south of 25° S.

- If $R = R^*$ is chosen as radius for the Web Mercator sphere, east-west calculations with spherical formulas give an error smaller than

$$\left| \frac{R^*}{a} \sqrt{1 - e^2 \tanh^2 \pi} - 1 \right| \doteq 0.444\%$$

North-south calculations give an error smaller than

$$\left| \frac{R^*}{a} \cdot \frac{1}{1 - e^2} - 1 \right| \doteq 0.561\%$$

This error is smaller than 0.5% north of 15° N and south of 15° S.

The radius may be decreased even further. Let R_m denote the radius, where the absolute relative deviation for the meridian on the equator equals the absolute relative deviation at $\text{gd}(\pi)$ and $\text{gd}(-\pi)$. Here, with (77) we have the equation

$$\begin{aligned} \frac{R_m}{a} \cdot \frac{1}{1-e^2} - 1 &= 1 - \frac{R_m}{a} \cdot \frac{(1-e^2 \tanh^2 \pi)^{\frac{3}{2}}}{1-e^2} \\ \Rightarrow \frac{R_m}{a} \cdot \frac{1 + (1-e^2 \tanh^2 \pi)^{\frac{3}{2}}}{1-e^2} &= 2 \\ \Rightarrow R_m &= a \cdot 2 \cdot \frac{1-e^2}{1 + (1-e^2 \tanh^2 \pi)^{\frac{3}{2}}} \doteq 6\,367\,117 \text{ m.} \end{aligned} \quad (79)$$

Figure 8 and Figure 9 also contain this case $R = R_m$.

- If $R = R_m$ is chosen as radius for the Web Mercator sphere, east-west calculations with spherical formulas give an error smaller than 0.5% between 80° S and 80° N, but slightly higher error values up to $\text{gd}(-\pi)$ south and $\text{gd}(\pi)$ north. North-south calculations give an error smaller than or equal 0.5%.

On small scale maps with Mercator projection (spherical or ellipsoidal alike), the scales on parallels in the north and south of the map will differ more from the scale in the center of the map than the deviations between the Web Mercator sphere and the WGS 84 ellipsoid. Therefore, any sphere radius $R \in [R_m, a]$ may be a valid solution for an application. www.netzwolf.info 2009 and Sturm 2020 select $R = R^*$ as “natural” choice (which is just a soft criterion). With it, errors are *mostly* lower than 0.5% (see above). Therefore, computed values with spherical formulas will have *usually* two significant digits. The exception is an equatorial area between 15° S and 15° N where the error rises up to 0.674%.

Another aspect of small scale maps is that the east-west relative deviation for a selected latitude stays constant independent from the range of longitude, but the north-south relative deviation is getting averaged. This is a further motivation not to optimize the peak north-south relative deviation too much in choosing R_m on cost of worsen the east-west relative deviation.

The results are illustrated with example maps given by Figure 10, Figure 11, Figure 12, and Figure 13. Here, $R = R^*$ is selected as radius for the Web Mercator sphere. The scale is computed according to the spherical model. The width of the map is stated for north, center, and south position, also the height of the map. All distances are computed with ellipsoidal and spherical formulas for comparison.

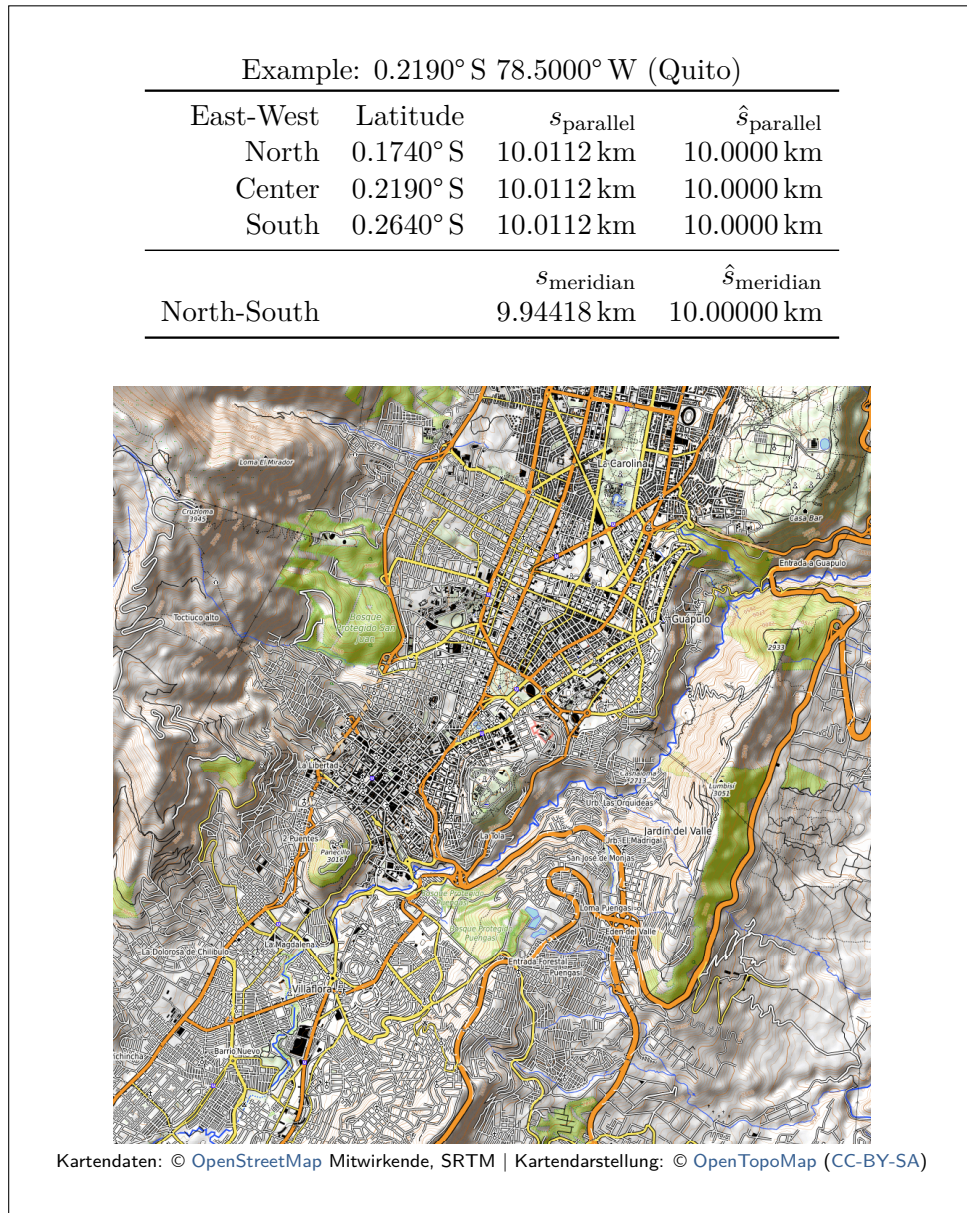


Figure 10: 10 cm by 10 cm map with a nominal (center) scale of 1:100 000. For this near-equator map a small east-west deviation (ca. 0.11%) and a large north-south deviation (ca. 0.56%) can be noted in concordance to Figure 8 and Figure 9.

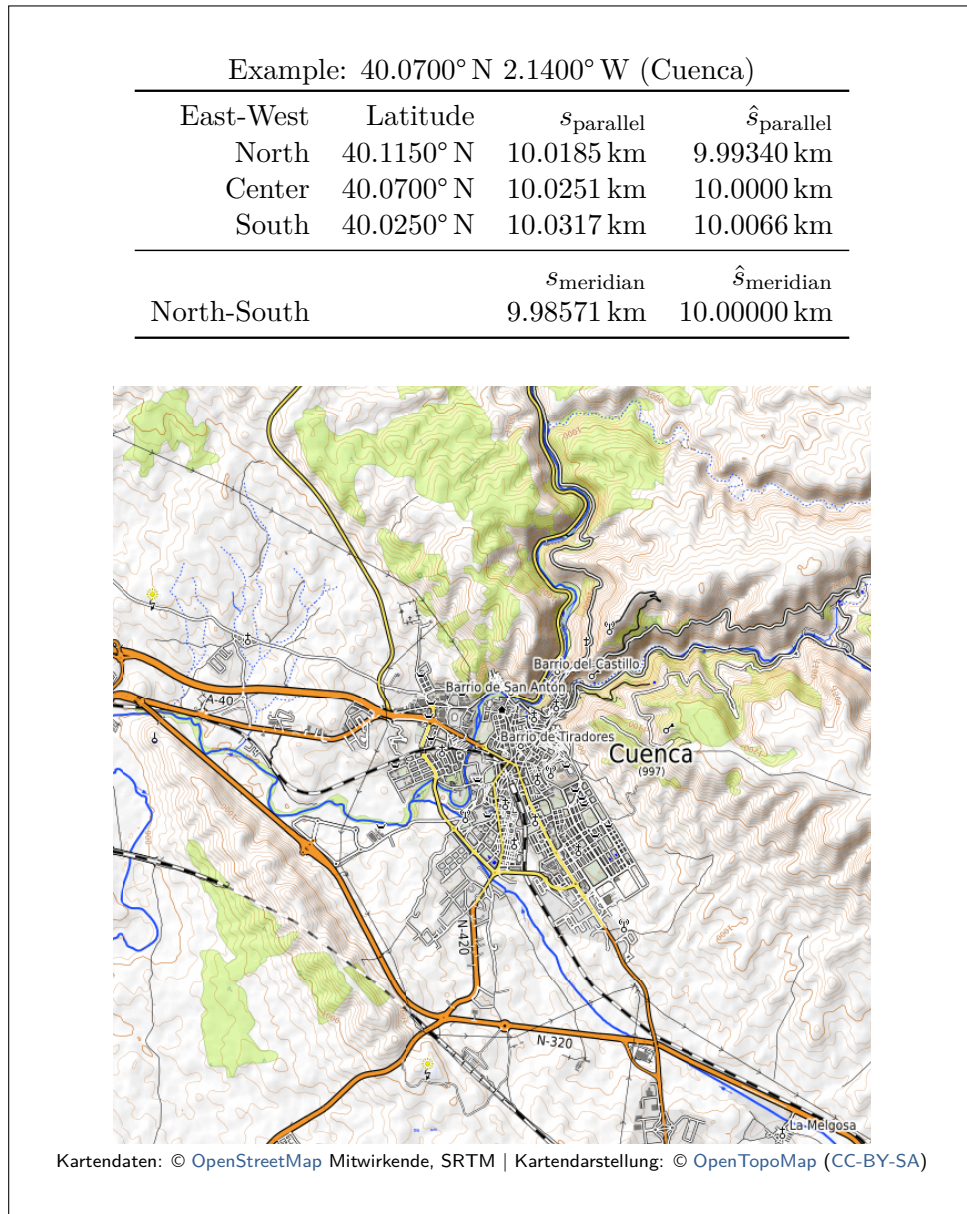


Figure 11: 10 cm by 10 cm map with a nominal (center) scale of 1:100 000. For this map near 40° N a small east-west deviation (ca. 0.25%) and a small north-south deviation (ca. 0.14%) can be noted in concordance to Figure 8 and Figure 9. The difference in width between North and South (ellipsoidal and spherical alike) is in about the same range.

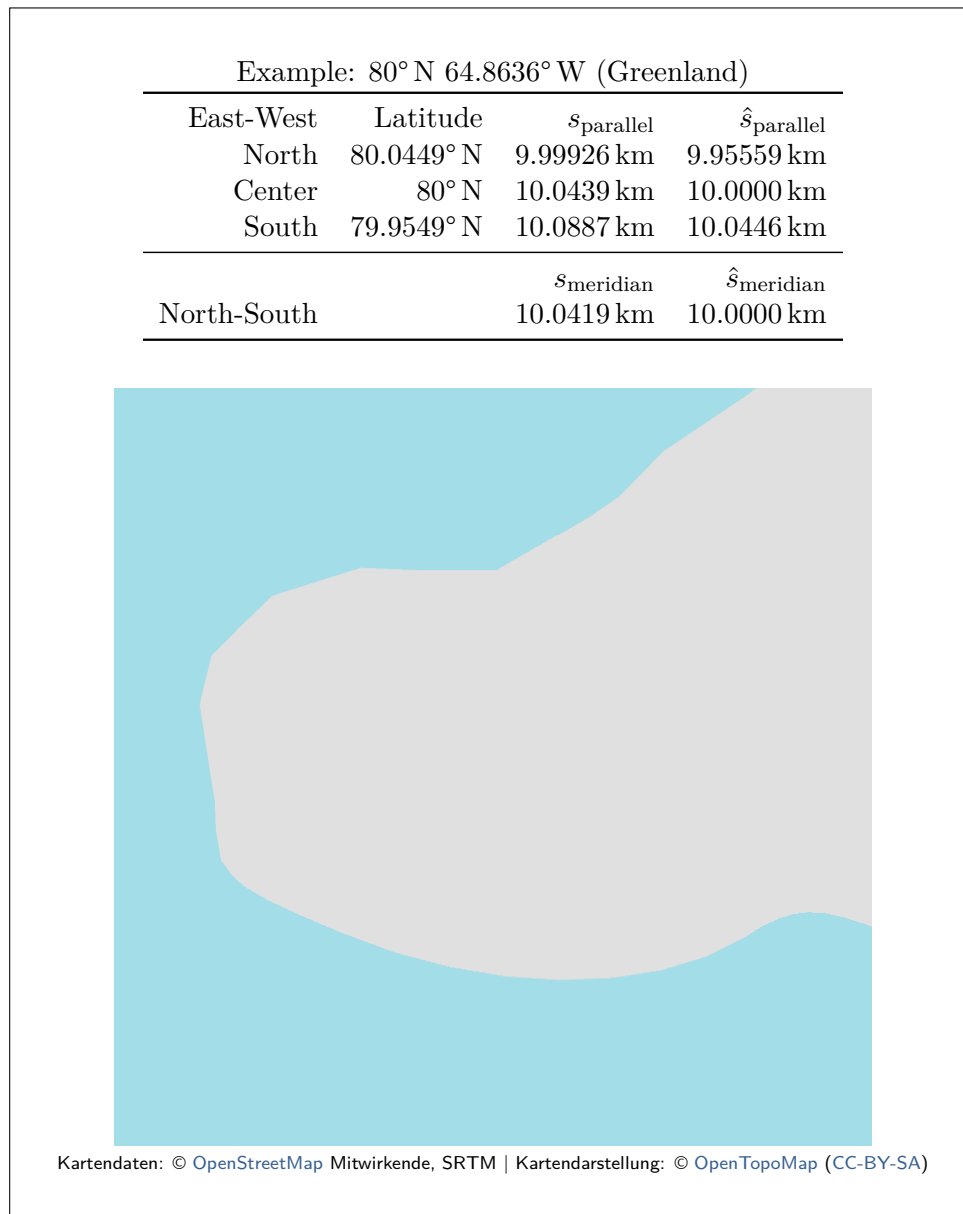


Figure 12: 10 cm by 10 cm map with a nominal (center) scale of 1:100 000.

For this map near 80° N a medium to large east-west deviation (ca. 0.44%) and a medium to large north-south deviation (ca. 0.42%) can be noted in concordance to Figure 8 and Figure 9. The difference in width between North and South (ellipsoidal and spherical alike) is in about twice in size.

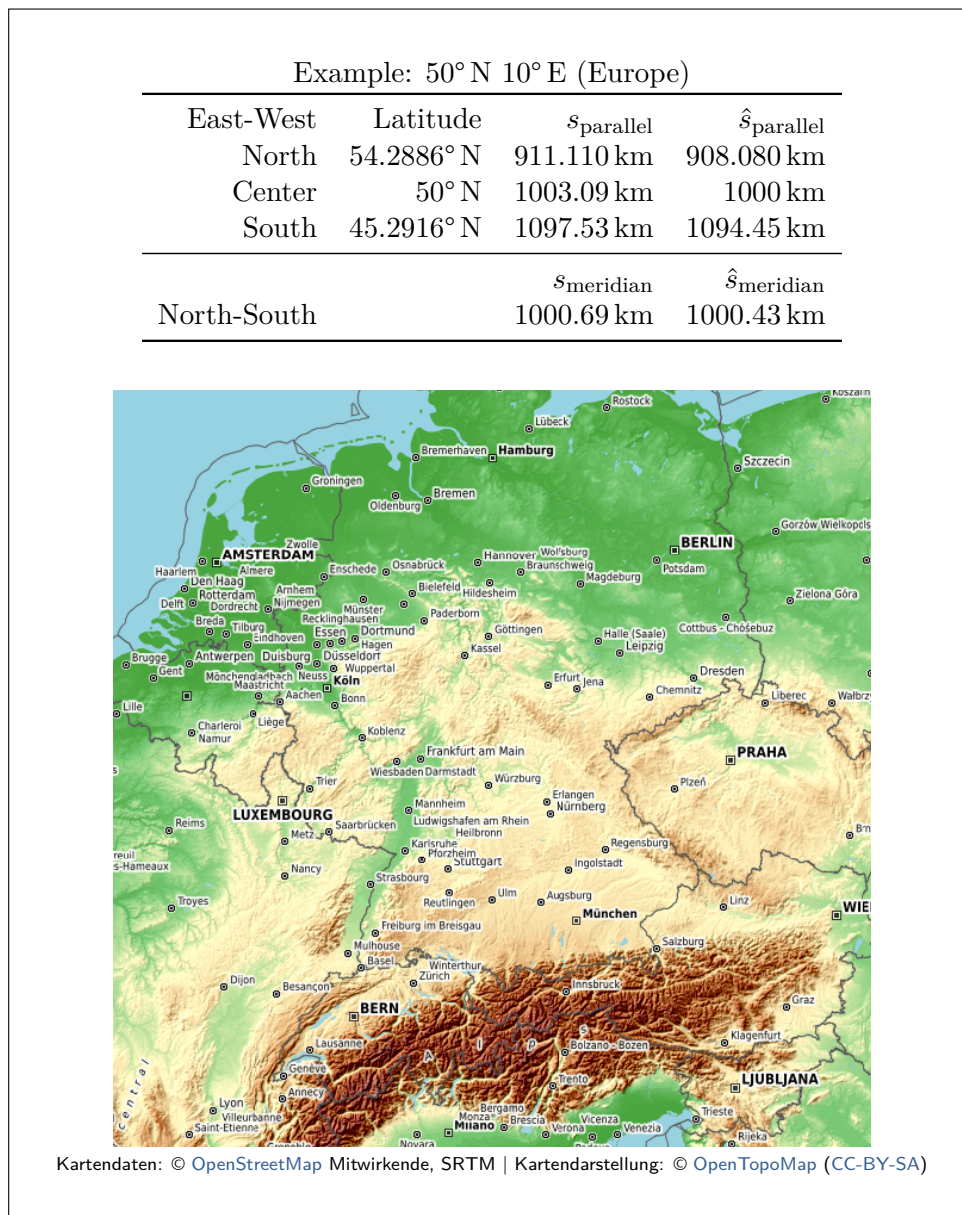


Figure 13: 10 cm by 10 cm map with a nominal (center) scale of 1:10 000 000. For this small scale map with center 50° N a small east-west deviation (from ca. 0.28% to ca. 0.33%) and a very small north-south deviation (ca. 0.026%) can be noted in concordance to Figure 8 and Figure 9. As Figure 9 shows, averaging near 50° N may give complete error extinction in lucky situations. The difference in width between North and South (ellipsoidal and spherical alike) is distinctly larger with 186 km in comparison to 1003 km respectively 1000 km center width.

4.3 Size of the Map Tiles

The equations for longitude range and latitude range for a map tile are easily seen by combining (43) with (42) and (41). With these values, size indications can be given by basic relations reprising (73) and (74):

Indications of size for map tile $T(Z, X, Y)$

Let zoom level $Z \in \mathbb{N}_0$, $N := 2^Z$, and let $X, Y \in \{0, N - 1\}$. Also, let a radius $R > 0$ be given. The spherical rectangle described by $M_U^{-1}(T(Z, X, Y)) = [\varphi_{\min}, \varphi_{\max}] \times [\lambda_{\min}, \lambda_{\max}]$ scaled with R , is given by

$$\begin{aligned} \text{height} &= R \cdot (\varphi_{\max} - \varphi_{\min}), \\ \text{north width} &= R \cdot (\lambda_{\max} - \lambda_{\min}) \cdot \cos \varphi_{\max}, \\ \text{south width} &= R \cdot (\lambda_{\max} - \lambda_{\min}) \cdot \cos \varphi_{\min}, \\ \text{area} &= R^2 \cdot (\lambda_{\max} - \lambda_{\min}) \cdot (\sin \varphi_{\max} - \sin \varphi_{\min}). \end{aligned}$$

Size comparison for $T(10, *, Y)$

The following table uses a zoom level 10, an arbitrary tile position X and some selected tile positions Y from north polar region to equator and a mean radius of $R^* = 6371$ km with the spherical model.

Y	south	north	height	width(S)	width(N)	area
0	85.021°	85.051°	3.383 km	3.393 km	3.372 km	11.4 km ²
127	79.171°	79.237°	7.322 km	7.344 km	7.300 km	53.6 km ²
255	66.513°	66.653°	15.536 km	15.580 km	15.492 km	241 km ²
433	26.431°	26.746°	34.958 km	35.006 km	34.910 km	1222 km ²
511	0.000°	0.352°	39.092 km	39.092 km	39.091 km	1528 km ²

The table above shows that for not-too-small zoom level values a perfect square map tile relates to a well-squarish region on the earth surface. But, as it is no surprise for the Mercator projection, the edge length for such a region shrinks from 39 km at the equator to about 15 km at the polar circles. About 50 percent of the unit square $[0, 1]^2$ for the presentable world lie beyond the polar circle where the edge length shrinks to nearly 3 km at the very end.

The logical consequence is that near the equator much higher zoom levels should be provided than in the polar regions to gain the same degree of resolution for the Earth surface. According to OpenStreetMap contributors 2019, OpenStreetMap and other providers for tile servers use a fixed range of zoom levels, sometimes more for national regions. Bundesamt für Kartographie und Geodäsie 2018 uses a staggered approach with zoom levels 0–9 for Earth, 10–14 for Europe, 15–18 for Germany.

5 Map Drawing

5.1 Covering a Map with Map Tiles

For applications like zoomable Web maps or static electronic or printed graphics, a rectangular selection of available map tiles is needed. A *map* is seen as a window showing some of the map tiles of the projected earth surface at a certain zoom level Z as displayed in Figure 14. Hereby, the window is completely covered with map tiles. For some special cases, some map tiles may appear more than once or may be virtual (not really existing).

Map description	
φ_1	latitude of the north side of the map
φ_2	latitude of the south side of the map with $\varphi_2 < \varphi_1$
λ_1	longitude of the west side of the map
λ_2	longitude of the east side of the map with $\lambda_1 < \lambda_2$
Z	zoom level with $Z \geq 0$
(X_1, Y_1)	map tile coordinate pair (north west)
(X_2, Y_2)	map tile coordinate pair (south east) with $X_2 \geq X_1, Y_2 \geq Y_1$
m_X	map window width given as multiplicity of map tiles with $m_X > 0$
m_Y	map window height given as multiplicity of map tiles with $m_Y > 0$
d_X	western offset given as multiplicity of map tiles with $0 \leq d_X < 1$
d_Y	northern offset given as multiplicity of map tiles with $0 \leq d_Y < 1$
\widehat{d}_Y	southern offset given as multiplicity of map tiles with $0 \leq \widehat{d}_Y < 1$

The map description above contains several redundant values which can be converted into each other as will be shown in the following. Also, we allow $Y_1 < 0$ and $Y_2 \geq 2^Z$ denoting virtual map tiles outside the unit square $[0, 1]^2$ described in Section 3. In the following, let $N := 2^Z$.

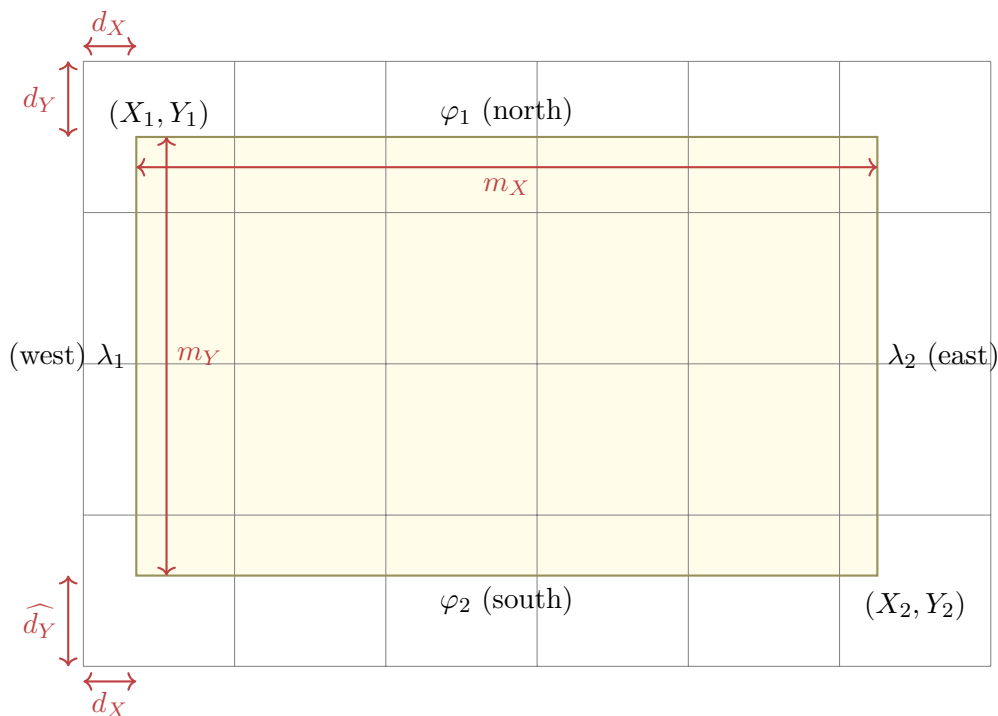


Figure 14: Map window which is covered by map tiles.

5.2 Map Dimensions from Latitude and Longitude Boundaries

If a zoom level Z and east, west, north, and south boundaries are given, the remaining map description parameters can be computed.

With (43), we have for all (x, y) inside a map tile $T(Z, X, Y)$ that

$$\frac{X}{N} \leq x \leq \frac{X+1}{N} \Leftrightarrow X \leq x \cdot N \leq X+1$$

and

$$\begin{aligned} \frac{N-Y-1}{N} \leq y \leq \frac{N-Y}{N} &\Leftrightarrow N-Y-1 \leq y \cdot N \leq N-Y \\ \Leftrightarrow Y-N \leq -y \cdot N \leq Y-N+1 &\Leftrightarrow Y \leq (1-y) \cdot N \leq Y+1 \end{aligned}$$

With this, we get:

Map computation from $Z, \varphi_1, \varphi_2, \lambda_1, \lambda_2$

Let $N := 2^Z$.

$$\begin{aligned} X_1 &= \lfloor x(\lambda_1) \cdot N \rfloor = \left\lfloor \left(\frac{\lambda_1}{2\pi} + \frac{1}{2} \right) \cdot 2^Z \right\rfloor = \left\lfloor (\lambda_1 + \pi) \cdot \frac{2^{Z-1}}{\pi} \right\rfloor, \\ Y_1 &= \lfloor (1-y(\varphi_1)) \cdot N \rfloor = \left\lfloor \left(\frac{1}{2} - \frac{\text{arggd } \varphi_1}{2\pi} \right) \cdot 2^Z \right\rfloor = \left\lfloor (\pi - \text{arggd } \varphi_1) \cdot \frac{2^{Z-1}}{\pi} \right\rfloor, \\ X_2 &= \lceil x(\lambda_2) \cdot N \rceil - 1 = \left\lceil \left(\frac{\lambda_2}{2\pi} + \frac{1}{2} \right) \cdot 2^Z \right\rceil - 1 = \left\lceil (\lambda_2 + \pi) \cdot \frac{2^{Z-1}}{\pi} \right\rceil - 1, \\ Y_2 &= \lceil (1-y(\varphi_2)) \cdot N \rceil - 1 = \left\lceil \left(\frac{1}{2} - \frac{\text{arggd } \varphi_2}{2\pi} \right) \cdot 2^Z \right\rceil - 1 \\ &= \left\lceil (\pi - \text{arggd } \varphi_2) \cdot \frac{2^{Z-1}}{\pi} \right\rceil - 1, \\ m_X &= (x(\lambda_2) - x(\lambda_1)) \cdot N = \frac{\lambda_2 - \lambda_1}{2\pi} \cdot 2^Z = (\lambda_2 - \lambda_1) \cdot \frac{2^{Z-1}}{\pi}, \\ m_Y &= (y(\varphi_1) - y(\varphi_2)) \cdot N = \frac{\text{arggd } \varphi_1 - \text{arggd } \varphi_2}{2\pi} \cdot 2^Z = (\text{arggd } \varphi_1 - \text{arggd } \varphi_2) \cdot \frac{2^{Z-1}}{\pi}, \\ d_X &= x(\lambda_1) \cdot N - X_1 = (\lambda_1 + \pi) \cdot \frac{2^{Z-1}}{\pi} - X_1, \\ d_Y &= (1-y(\varphi_1)) \cdot N - Y_1 = (\pi - \text{arggd } \varphi_1) \cdot \frac{2^{Z-1}}{\pi} - Y_1, \\ \widehat{d}_Y &= Y_2 - (1-y(\varphi_2)) \cdot N + 1 = Y_2 - (\pi - \text{arggd } \varphi_2) \cdot \frac{2^{Z-1}}{\pi} + 1 \\ &= Y_2 - Y_1 + 1 - m_Y - d_Y. \end{aligned}$$

For a more concise representation, we define some help variables $a_1, a_2, b_1,$ and b_2 . They are the projected and scaled boundaries.

Map computation from $Z, \varphi_1, \varphi_2, \lambda_1, \lambda_2$ (version 2)

$$\begin{aligned}
P &:= \frac{2^{Z-1}}{\pi}, \\
a_1 &:= (\lambda_1 + \pi) \cdot P, \\
b_1 &:= (\pi - \text{arggd } \varphi_1) \cdot P, \\
a_2 &:= (\lambda_2 + \pi) \cdot P, \\
b_2 &:= (\pi - \text{arggd } \varphi_2) \cdot P, \\
X_1 &= \lfloor a_1 \rfloor, \\
Y_1 &= \lfloor b_1 \rfloor, \\
X_2 &= \lceil a_2 \rceil - 1, \\
Y_2 &= \lceil b_2 \rceil - 1, \\
m_X &= a_2 - a_1, \\
m_Y &= b_2 - b_1, \\
d_X &= a_1 - X_1, \\
d_Y &= b_1 - Y_1, \\
\widehat{d}_Y &= Y_2 - b_2 + 1.
\end{aligned}$$

The auxiliary variables (a_1, b_1) and (a_2, b_2) can be interpreted as coordinate pairs from a rescaled coordinate system where the unit square $[0, 1]^2$ is projected to $[0, 2^Z]^2$. Note that the y coordinate was scaled negative (flipped). Here, each map tile has edge length 1.

5.3 Map Dimensions for a fixed Map Size from a Center Position

If a zoom level Z and a fixed map size m_X and m_Y are given, the remaining map description parameters can be computed from a given center position (φ_c, λ_c) .

The computation formulas of Section 5.2 are complemented with help variables a_c and b_c for the projected and scaled center position. The remaining parameters can be resolved as follows:

Map computation from $Z, m_X, m_Y, \varphi_c, \lambda_c$

$$\begin{aligned}
 P &:= \frac{2^{Z-1}}{\pi}, \\
 a_c &:= (\lambda_c + \pi) \cdot P, \\
 b_c &:= (\pi - \operatorname{arggd} \varphi_c) \cdot P, \\
 a_1 &:= a_c - \frac{1}{2}m_X, \\
 b_1 &:= b_c - \frac{1}{2}m_Y, \\
 a_2 &:= a_c + \frac{1}{2}m_X, \\
 b_2 &:= b_c + \frac{1}{2}m_Y, \\
 \lambda_1 &= \frac{a_1}{P} - \pi, \\
 \varphi_1 &= \operatorname{gd} \left(\pi - \frac{b_1}{P} \right), \\
 \lambda_2 &= \frac{a_2}{P} - \pi, \\
 \varphi_2 &= \operatorname{gd} \left(\pi - \frac{b_2}{P} \right), \\
 X_1 &= \lfloor a_1 \rfloor, \\
 Y_1 &= \lfloor b_1 \rfloor, \\
 X_2 &= \lceil a_2 \rceil - 1, \\
 Y_2 &= \lceil b_2 \rceil - 1, \\
 d_X &= a_1 - X_1, \\
 d_Y &= b_1 - Y_1, \\
 \widehat{d}_Y &= Y_2 - b_2 + 1.
 \end{aligned}$$

Note that this construction from a center position can easily be adapted to a construction from other reference positions. Candidates are map edge positions like west, northwest, north, etc.

5.4 Map Dimensions for a fixed Map Size to fit Latitude and Longitude Boundaries

If a fixed map size m_X and m_Y is given and the map should fit an area described by latitude and longitude boundaries $\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\lambda}_1, \tilde{\lambda}_2$, the minimum zoom level and the remaining map description parameters can be computed.

Basically, the zoom level Z is computed and the area is centered inside the map window.

From Section 5.2, we have

$$m_X = (\lambda_2 - \lambda_1) \cdot P, \quad m_Y = (\text{arggd } \varphi_1 - \text{arggd } \varphi_2) \cdot P, \quad Z = 1 + \log_2(\pi \cdot P)$$

A fitting zoom level is found, if the lower P from the first two equations inserted with $\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\lambda}_1, \tilde{\lambda}_2$ is taken and Z is computed from it:

Map computation from $m_X, m_Y, \tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\lambda}_1, \tilde{\lambda}_2$

$$\begin{aligned} Z &= \max \left\{ 0, 1 + \left\lceil \log_2 \left(\pi \cdot \min \left\{ \frac{m_X}{\tilde{\lambda}_2 - \tilde{\lambda}_1}, \frac{m_Y}{\text{arggd } \tilde{\varphi}_1 - \text{arggd } \tilde{\varphi}_2} \right\} \right) \right\rceil \right\}, \\ P &:= \frac{2^{Z-1}}{\pi}, \\ a_c &:= \left(\frac{\tilde{\lambda}_1 + \tilde{\lambda}_2}{2} + \pi \right) \cdot P, \\ b_c &:= \left(\pi - \frac{\text{arggd } \tilde{\varphi}_1 + \text{arggd } \tilde{\varphi}_2}{2} \right) \cdot P, \\ a_1 &:= a_c - \frac{1}{2} m_X, \\ b_1 &:= b_c - \frac{1}{2} m_Y, \\ a_2 &:= a_c + \frac{1}{2} m_X, \\ b_2 &:= b_c + \frac{1}{2} m_Y, \\ \lambda_1 &= \frac{a_1}{P} - \pi, \\ \varphi_1 &= \text{gd} \left(\pi - \frac{b_1}{P} \right), \\ \lambda_2 &= \frac{a_2}{P} - \pi, \\ \varphi_2 &= \text{gd} \left(\pi - \frac{b_2}{P} \right), \\ X_1 &= \lfloor a_1 \rfloor, \\ Y_1 &= \lfloor b_1 \rfloor, \\ X_2 &= \lceil a_2 \rceil - 1, \\ Y_2 &= \lceil b_2 \rceil - 1, \\ d_X &= a_1 - X_1, \\ d_Y &= b_1 - Y_1, \\ \widehat{d}_Y &= Y_2 - b_2 + 1. \end{aligned}$$

5.5 Application (Target) Coordinate System

The final step for map drawing is fitting to the target coordinate system of the application. Depending on the constraints of the application, the target system has positive orientation (e.g. TikZ for L^AT_EX) or negative orientation (many computer screen drawing libraries from programming languages), see Figure 15.

In the following, without loss of generality, the origin of the target system is assumed to be the southwest or northwest corner of the map, see Figure 15. Additional to a map description as in Section 5.1, an edge length for a single square tile is needed for computation. Let s_T be this tile size (edge length). For a computer screen application $s_T = 256$ is the natural choice, if the pixel file for a tile contains 256 times 256 pixels. With this data and (37), (38), (39), the fitting to the target coordinate system is simple scaling and shifting.

Projection to the application coordinate system

Consider a map description from Section 5.1. Let s_T denote the size (edge length) of a single tile. Also define

$$s_M := s_T \cdot N = s_T \cdot 2^Z.$$

The projection of a pair (φ, λ) consisting of a latitude angle φ and a longitude angle λ to a pair (u_x, u_y) of Cartesian coordinates is given as follows.

- Coordinate system with positive orientation:

$$\begin{aligned} u_x &= s_M \cdot (x(\lambda) - x(\lambda_1)) = \frac{s_M}{2\pi} \cdot (\lambda - \lambda_1), \\ u_y &= s_M \cdot (y(\varphi) - y(\varphi_2)) = \frac{s_M}{2\pi} \cdot (\text{arggd}(\varphi) - \text{arggd}(\varphi_2)) \\ &= \frac{s_M}{2\pi} \cdot \ln \left(\frac{\tan \left(\frac{1}{2}\varphi + \frac{1}{4}\pi \right)}{\tan \left(\frac{1}{2}\varphi_2 + \frac{1}{4}\pi \right)} \right). \end{aligned}$$

- Coordinate system with negative orientation:

$$\begin{aligned} u_x &= s_M \cdot (x(\lambda) - x(\lambda_1)) = \frac{s_M}{2\pi} \cdot (\lambda - \lambda_1), \\ u_y &= s_M \cdot (y(\varphi_1) - y(\varphi)) = \frac{s_M}{2\pi} \cdot (\text{arggd}(\varphi_1) - \text{arggd}(\varphi)) \\ &= \frac{s_M}{2\pi} \cdot \ln \left(\frac{\tan \left(\frac{1}{2}\varphi_1 + \frac{1}{4}\pi \right)}{\tan \left(\frac{1}{2}\varphi + \frac{1}{4}\pi \right)} \right). \end{aligned}$$

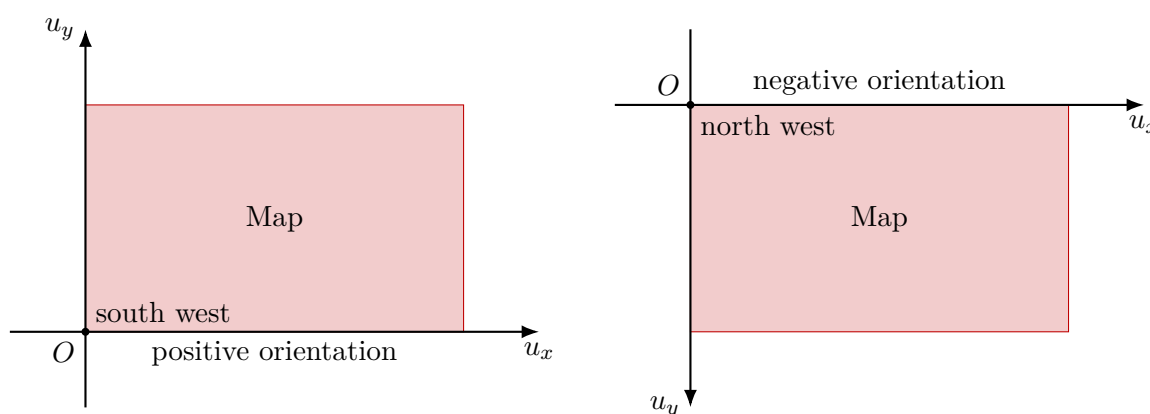


Figure 15: Orientation of the application coordinate system

The calculations above are found straight forward using (38) and (39) together with the representation (29) of the inverse Gudermann function.

This gives the projection of (φ, λ) to a target coordinate pair (u_x, u_y) on a computer screen or another application image. The target map is covered with tiles $T(Z, X, Y)$ as described in Section 5.1. The center position (x, y) of a tile $T(Z, X, Y)$ in unified Cartesian coordinates is given by (43) as

$$x = \frac{X + \frac{1}{2}}{N}, \quad y = \frac{N - Y - \frac{1}{2}}{N}.$$

The positioning of these tiles in the application coordinate system is presented by the following.

Map tile positioning in the application coordinate system

Consider a map description from Section 5.1. Let s_T denote the size (edge length) of a single tile. Also define

$$s_M := s_T \cdot N = s_T \cdot 2^Z.$$

The center position (u_x, u_y) of a tile $T(Z, X, Y)$ with

$$X_1 \leq X \leq X_2, \quad Y_1 \leq Y \leq Y_2$$

in the application coordinate system is given as follows.

- Coordinate system with positive orientation:

$$\begin{aligned} u_x &= s_M \cdot \left(\frac{X + \frac{1}{2}}{N} - x(\lambda_1) \right) \\ &= s_T \cdot (X - X_1) + s_T \cdot \left(\frac{1}{2} - d_X \right), \\ u_y &= s_M \cdot \left(\frac{N - Y - \frac{1}{2}}{N} - y(\varphi_2) \right) \\ &= s_T \cdot (Y_2 - Y) + s_T \cdot \left(\frac{1}{2} - \widehat{d}_Y \right). \end{aligned}$$

- Coordinate system with negative orientation:

$$\begin{aligned} u_x &= s_M \cdot \left(\frac{X + \frac{1}{2}}{N} - x(\lambda_1) \right) \\ &= s_T \cdot (X - X_1) + s_T \cdot \left(\frac{1}{2} - d_X \right), \\ u_y &= s_M \cdot \left(y(\varphi_1) - \frac{N - Y - \frac{1}{2}}{N} \right) \\ &= s_T \cdot (Y - Y_1) + s_T \cdot \left(\frac{1}{2} - d_Y \right). \end{aligned}$$

For the offset values d_X , \widehat{d}_Y , and d_Y consult Figure 14 and Section 5.1.

6 Excursion: Loxodromic and orthodromic Distance

6.1 Loxodromic Distance

The arc length s of a loxodrome on a unit sphere which is parameterized by latitude following (11) is described by (9) as

$$\frac{ds}{d\varphi}(\varphi) = \frac{d\tilde{\varphi}^{-1}}{d\varphi}(\varphi) = \frac{1}{\cos \beta}$$

which gives through integration

$$s(\varphi) = \frac{\varphi}{\cos \beta} + s_0. \tag{80}$$

Now, we consider a loxodrome running from the position (φ_1, λ_1) to (φ_2, λ_2) . The non-negative length s of the loxodrome piece between these two positions following (11) and (80) is given by

$$s = \frac{|\varphi_2 - \varphi_1|}{\cos \beta} \quad \text{with} \quad \tan \beta = \frac{\lambda_2 - \lambda_1}{\text{arggd } \varphi_2 - \text{arggd } \varphi_1}. \tag{81}$$

Since $\frac{1}{\cos \beta} = \sqrt{1 + \tan^2 \beta}$ for $|\beta| < \frac{\pi}{2}$, this gives

$$s = |\varphi_2 - \varphi_1| \cdot \sqrt{1 + \left(\frac{\lambda_2 - \lambda_1}{\text{arggd } \varphi_2 - \text{arggd } \varphi_1} \right)^2}, \quad \text{for } \varphi_1 \neq \varphi_2. \tag{82}$$

If $\varphi_1 = \varphi_2$, we run on a circle of latitude and we get directly

$$s = |\lambda_2 - \lambda_1| \cdot \cos(\varphi_1), \quad \text{for } \varphi_1 = \varphi_2. \tag{83}$$

With the abbreviations $\Delta\varphi := \varphi_2 - \varphi_1$ and $\Delta\lambda := \lambda_2 - \lambda_1$, we finally consider an approximation formula for (82), if $\Delta\varphi$ is very small. The derivatives of $\text{arggd } x$ are with (16) given by

$$\begin{aligned} \frac{d}{dx} \text{arggd}(x) &= \frac{1}{\cos x}, \\ \frac{d^2}{dx^2} \text{arggd}(x) &= \frac{\tan x}{\cos x}, \\ \frac{d^3}{dx^3} \text{arggd}(x) &= \frac{\cos x \cdot (1 + \tan^2 x) - \tan x \cdot (-\sin x)}{\cos^2 x} = \frac{1 + 2 \tan^2 x}{\cos x}. \end{aligned}$$

The Taylor approximation for arggd is found as

$$\begin{aligned} \text{arggd}(\varphi_2) &= \text{arggd}(\varphi_1) + \frac{d}{d\varphi} \text{arggd}(\varphi_1) \cdot \Delta\varphi + \frac{1}{2} \frac{d^2}{d\varphi^2} \text{arggd}(\varphi_1) \cdot \Delta\varphi^2 \\ &\quad + \frac{1}{6} \frac{d^3}{d\varphi^3} \text{arggd}(\varphi_1) \cdot \Delta\varphi^3 + \dots \\ &= \text{arggd}(\varphi_1) + \frac{1}{\cos \varphi_1} \cdot \Delta\varphi + \frac{1}{2} \frac{\tan \varphi_1}{\cos \varphi_1} \cdot \Delta\varphi^2 + \frac{1}{6} \frac{1 + 2 \tan^2 \varphi_1}{\cos \varphi_1} \cdot \Delta\varphi^3 + \dots \end{aligned}$$

We put this into (82) and get

$$\begin{aligned}
 s &= \sqrt{(\varphi_2 - \varphi_1)^2 + \left(\frac{\lambda_2 - \lambda_1}{\frac{\operatorname{arggd} \varphi_2 - \operatorname{arggd} \varphi_1}{\varphi_2 - \varphi_1}}\right)^2} \\
 &\approx \sqrt{\Delta\varphi^2 + \left(\frac{\Delta\lambda}{\frac{1}{\cos \varphi_1} + \frac{1}{2} \frac{\tan \varphi_1}{\cos \varphi_1} \cdot \Delta\varphi + \frac{1}{6} \frac{1+2 \tan^2 \varphi_1}{\cos \varphi_1} \cdot \Delta\varphi^2}\right)^2} \\
 &= \sqrt{\Delta\varphi^2 + \left(\frac{\Delta\lambda \cdot \cos \varphi_1}{1 + \frac{1}{2} \tan \varphi_1 \cdot \Delta\varphi + \frac{1}{6}(1 + 2 \tan^2 \varphi_1) \cdot \Delta\varphi^2}\right)^2}.
 \end{aligned}$$

Loxodromic Distance between (φ_1, λ_1) and (φ_2, λ_2)

Let (φ_1, λ_1) and (φ_2, λ_2) denote two positions on the unit sphere. Let $\Delta\varphi := \varphi_2 - \varphi_1$ and $\Delta\lambda := \lambda_2 - \lambda_1$. Further, consider an $\varepsilon > 0$ near zero.

The loxodromic distance s between these positions is given by:

$$s = |\Delta\varphi| \cdot \sqrt{1 + \left(\frac{\Delta\lambda}{\operatorname{arggd} \varphi_2 - \operatorname{arggd} \varphi_1}\right)^2}, \quad \text{for } |\Delta\varphi| \geq \varepsilon. \tag{84}$$

$$s \approx \sqrt{\Delta\varphi^2 + \left(\frac{\Delta\lambda \cdot \cos \varphi_1}{1 + \frac{1}{2} \tan \varphi_1 \cdot \Delta\varphi + \frac{1}{6}(1 + 2 \tan^2 \varphi_1) \cdot \Delta\varphi^2}\right)^2}, \quad \text{for } |\Delta\varphi| < \varepsilon. \tag{85}$$

- For Pseudo-Mercator on a sphere, this distance has to be multiplied with a radius R , e.g. $R = R^* = 6\,371\,000$ m. An example is given in Figure 17.
- (84) and (85) compute the distance faithful to the given latitude and longitude values. On a globe, there is a second loxodrome which is shorter, if $|\Delta\lambda| > \pi$.

6.2 Length of orthodromic Distance

Now, we consider an orthodrome running from the position (φ_1, λ_1) to (φ_2, λ_2) . The orthodrome is a great-circle through

$$\vec{a} := \vec{r}(\varphi_1, \lambda_1) = \begin{pmatrix} \cos \lambda_1 \cdot \cos \varphi_1 \\ \sin \lambda_1 \cdot \cos \varphi_1 \\ \sin \varphi_1 \end{pmatrix} \quad \text{and} \quad \vec{b} := \vec{r}(\varphi_2, \lambda_2) = \begin{pmatrix} \cos \lambda_2 \cdot \cos \varphi_2 \\ \sin \lambda_2 \cdot \cos \varphi_2 \\ \sin \varphi_2 \end{pmatrix}$$

on our unit sphere with the same center as the sphere, see Figure 16.

The central angle ψ (in radians) between the vectors \vec{a} and \vec{b} denoting the two considered points on the sphere is identical to the orthodromic distance (arc distance) between these points. It can be computed via the scalar product formula (note that $\|\vec{a}\| = \|\vec{b}\| = 1$):

$$\begin{aligned}
 \cos \psi &= \langle \vec{a} | \vec{b} \rangle = a_x b_x + a_y b_y + a_z b_z \\
 &= \cos \lambda_1 \cdot \cos \varphi_1 \cdot \cos \lambda_2 \cdot \cos \varphi_2 + \sin \lambda_1 \cdot \cos \varphi_1 \cdot \sin \lambda_2 \cdot \cos \varphi_2 + \sin \varphi_1 \cdot \sin \varphi_2 \\
 &= \sin \varphi_1 \cdot \sin \varphi_2 + \cos \varphi_1 \cdot \cos \varphi_2 \cdot \cos(\lambda_2 - \lambda_1).
 \end{aligned}$$

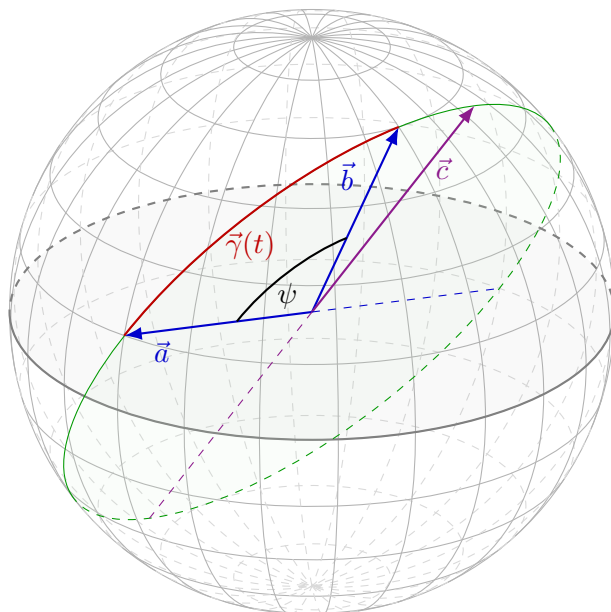


Figure 16: Orthodrome as great-circle through \vec{a} and \vec{b} . The central angle ψ in radians is equal to the orthodromic distance between \vec{a} and \vec{b} . \vec{c} is orthogonal to \vec{a} .

Orthodromic Distance between (φ_1, λ_1) and (φ_2, λ_2)

Let (φ_1, λ_1) and (φ_2, λ_2) denote two positions on the unit sphere. The orthodromic distance ψ between these positions is given by:

$$\psi = \arccos(\sin \varphi_1 \cdot \sin \varphi_2 + \cos \varphi_1 \cdot \cos \varphi_2 \cdot \cos(\lambda_2 - \lambda_1)) \in [0, \pi] \tag{86}$$

- For Pseudo-Mercator on a sphere, this distance (86) has to be multiplied with a radius R , e.g. $R = R^* = 6\,371\,000$ m. An example is given in Figure 17.

The vectors \vec{a} and \vec{b} span a plane which intersects the sphere in the loxodrome. To construct the loxodrome or the loxodrome piece between \vec{a} and \vec{b} , we consider a third unit vector \vec{c} inside this plane which is constructed to be orthogonal to \vec{a} .

$$\vec{c} := \vec{b} - \langle \vec{a} | \vec{b} \rangle \cdot \vec{a} = \vec{b} - \cos \psi \cdot \vec{a}$$

is orthogonal to \vec{a} , but with length

$$\begin{aligned} \|\vec{c}\| &= \sqrt{\langle \vec{b} - \cos \psi \cdot \vec{a} | \vec{b} - \cos \psi \cdot \vec{a} \rangle} \\ &= \sqrt{\langle \vec{b} | \vec{b} \rangle - 2 \cos \psi \cdot \langle \vec{a} | \vec{b} \rangle + \cos^2 \psi \cdot \langle \vec{a} | \vec{a} \rangle} \\ &= \sqrt{1 - 2 \cos \psi \cdot \cos \psi + \cos^2 \psi} \\ &= \sqrt{\sin^2 \psi} = \sin \psi, \quad \text{since } \psi \in [0, \pi]. \end{aligned}$$

Finally, the unit vector \vec{c} is found as

$$\vec{c} := \frac{\vec{c}}{\|\vec{c}\|} = \frac{\vec{b} - \cos \psi \cdot \vec{a}}{\sin \psi}. \tag{87}$$

The great-circle through \vec{a} and \vec{b} is a unit circle for the two-dimensional coordinate system given by the orthogonal unit vectors \vec{a} and \vec{c} , also see Figure 16. This results in the following orthodrome representation.



Underlying map © Bundesamt für Kartographie und Geodäsie 2020, Datenquellen

Figure 17: Distance between Munich (48.14° N 11.58° E) and Los Angeles (34.05° N 118.24° W). The (spherical) loxodromic distance computed according to (84) with $R = R^* = 6\,371\,000\text{ m}$ yields 10 922 198 m. The (spherical) orthodromic distance computed according to (86) with the identical radius gives 9 606 242 m. For comparison, also the geodesic distance for the WGS 84 ellipsoid is computed with GeographicLib (Karney 2019) as 9 628 852 m.

Orthodrome representation

An orthodrome piece connecting two positions (φ_1, λ_1) and (φ_2, λ_2) on the unit sphere has a representation

$$\vec{\gamma} : [0, 1] \rightarrow \mathbb{R}^3, \quad t \mapsto \vec{\gamma}(t) = \cos(t\psi) \cdot \vec{a} + \sin(t\psi) \cdot \vec{c} \tag{88}$$

$$= \frac{\sin((1-t)\psi)}{\sin \psi} \cdot \vec{a} + \frac{\sin(t\psi)}{\sin \psi} \cdot \vec{b}, \tag{89}$$

where

$$\psi = \arccos(\sin \varphi_1 \cdot \sin \varphi_2 + \cos \varphi_1 \cdot \cos \varphi_2 \cdot \cos(\lambda_2 - \lambda_1)),$$

$$\vec{a} = \begin{pmatrix} \cos \lambda_1 \cdot \cos \varphi_1 \\ \sin \lambda_1 \cdot \cos \varphi_1 \\ \sin \varphi_1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} \cos \lambda_2 \cdot \cos \varphi_2 \\ \sin \lambda_2 \cdot \cos \varphi_2 \\ \sin \varphi_2 \end{pmatrix}, \quad \vec{c} = \frac{\vec{b} - \cos \psi \cdot \vec{a}}{\sin \psi}.$$

It holds

$$\vec{\gamma}(0) = \vec{a}, \quad \vec{\gamma}(1) = \vec{b}.$$

For one point of the loxodrome with coordinates x, y, z the latitude φ and the longitude λ is computed by

$$\varphi = \arcsin z, \tag{90}$$

$$\lambda = \text{sgn}(y) \cdot \arccos \frac{x}{\sqrt{x^2 + y^2}} = \text{sgn}(y) \cdot \arccos \frac{x}{\cos \varphi} \tag{91}$$

With

$$\vec{\gamma}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

we get from (88) for the first coordinate

$$\begin{aligned} x(t) &= \cos(t\psi) \cdot a_x + \sin(t\psi) \cdot c_x \\ &= \cos(t\psi) \cdot a_x + \sin(t\psi) \cdot \frac{b_x - \cos \psi \cdot a_x}{\sin \psi} \\ &= \frac{\sin \psi \cdot \cos(t\psi) \cdot a_x - \cos \psi \cdot \sin(t\psi) \cdot a_x + \sin(t\psi) \cdot b_x}{\sin \psi} \\ &= \frac{\sin((1-t)\psi) \cdot a_x + \sin(t\psi) \cdot b_x}{\sin \psi}. \end{aligned}$$

This is the first coordinate of (89). The other coordinates are seen analogously. (90) and (91) are found by dissolving (1).

7 Conclusion

Web Mercator maps supplied by Web tile servers opened a door to general purpose maps for everyone. Despite the lack of accurateness in relation to spherical projection for ellipsoidal data, the advantages of public access and simplified computation predominate.

The classical spherical Mercator Projection (30), (31), (32) derived from the construction of a loxodrome (11) is adapted to a unified clipped projection (37), (38), (39) to the unit square $[0, 1]^2$. This projection uses the Gudermann function and its inverse which are given in many shapes by (13) to (29).

The unit square is disassembled into map tiles (43) which correspond to pixel graphics accessible over the Web (OpenStreetMap contributors 2019). For specific map applications, such map tiles are compound into final maps (Section 5). On such maps, loxodromic distances (84), (85) and orthodromic distances (86) can be depicted.

The price for using a spherical projection is a certain deviation from the “real world” taken as WGS 84 ellipsoid. Several aspects of this error are discussed in Section 4 with a choice (75) for the earth radius as $R^* := 6\,371\,000$ m. Errors may rise up to 0.674%, but are usually lower than 0.5% giving two significant digits.

The work described here involves derivation of classical Mercator formulas and application for Web Mercator tiles. All computations are presented in detail with the idea of algorithmic implementation.

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