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Relational Mathematics Continued

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Abstract

This is in some sense an addendum to [Sch11]. It originated from work on diverse other topics during which a lot of purely relational results with broad applicability have been produced. These include results on domain construction with novel formulae for existential and inverse image, a relational calculus for binary mappings, and the development of a formally derived relational calculus of Kronecker-, strict fork-, and strict join-operators. The many visualizations in this report make it also a scrap- and picture book for examples.

Keywords relational mathematics, relation algebra, domain construction, vectorization, binary mapping, Kronecker-, fork-, and join-operator, products, existential and inverse image

1 Introduction

In this report, several definitions, propositions and constructions are collected that already would have been incorporated in the book [Sch11] when they had been available at that time. This work is completely based on relation-algebraic methods. Nevertheless, we often use terms such as *set, powerset, etc.* to give intuition for the concepts intended.

Included is in Chapt. 2, what has to be mentioned from known relational methods to make the article self-contained. In addition, several new ideas of this kind are elaborated. Then follows a further study of the membership relation in Chapt. 3 and, based on it, a presentation of novel insights on existential and inverse images in Chapt. 5. To underpin the often quite intuitive formulae with rigorous relation-algebraic proofs for the first time turned out to be an unexpectedly difficult task.

The categorical product is again studied in Chapt. 6 and in Chapt. 7, working also with vectorization. Therewith, a relation may be seen in different incarnations, a relation as a rectangular Boolean matrix or a Boolean vector along the powerset, offering intricate interdependencies.

While relations lend themselves mainly to being studied with linear concepts, it also possible to approach binary mappings or operations via relational mathematics as in Chapt. 8. Application of such concepts allows to study Boolean algebra from quite a different perspective in Chapt. 9.

A slight generalization has taken place: It is known that relational mathematics admits also non-representable relation algebras as models — in case the Point Axiom should not have been postulated.

The presentation via computer-generated examples allows a very detailed view. They have been generated with the language TituRel (see [Sch04]), that directly interprets relational terms and formulae. So one can be sure to see the results of the explanations in the text directly mirrored. Accumulating such a multitude of rules and formulae follows the idea of René Descartes, who is told to have said: „Jedes Problem, das ich gelöst hatte, wurde zu einer Regel, mit deren Hilfe später weitere Probleme gelöst werden konnten.“

2 Prerequisites

The prerequisites presented routinely for relational work are fairly well-known: We will work with heterogeneous relations and provide a general reference to [Sch11], but also to the earlier [SS89, SS93, SHW97]. Our operations are, thus, binary union "$\cup$", intersection "$\cap$", composition "$;$", unary negation "$\overline{}$", transposition or conversion "$^{\mathsf{T}}$", together with zero-ary null relations "$\perp\!\!\!\perp$", universal relations "$\top\!\!\!\top$", and identities "$\mathbb{I}$". A *heterogeneous relation algebra*

- is a category wrt. composition "$;$" and identities $\mathbb{I}$,
- has as morphism sets complete atomic boolean lattices with $\cup, \cap, \overline{}, \perp\!\!\!\perp, \top\!\!\!\top, \subseteq$,
- obeys rules for transposition $^{\mathsf{T}}$ in connection with the latter two concepts that may be stated in either one of the following two ways:

  Dedekind rule:

  $R;S \cap Q \subseteq (R \cap Q;S^{\mathsf{T}});(S \cap R^{\mathsf{T}};Q)$

  Schröder equivalences:

  $A;B \subseteq C \iff A^{\mathsf{T}};\overline{C} \subseteq \overline{B} \iff \overline{C};B^{\mathsf{T}} \subseteq \overline{A}$

The two rules are equivalent in the context mentioned. Many rules follow out of this setting; not least everything for the concepts of a function, mapping, or ordering; e.g., that mappings f may be *shunted*, i.e., that $A;f \subseteq B \iff A \subseteq B;f^{\mathsf{T}}$. The rule $(A \cap B;g^{\mathsf{T}});g = A;g \cap B$ for univalent g is also frequently applied and sometimes referred to as *destroy and append*; Prop. 5.4 of [Sch11].

A new and widely useful rule serves to negate the left-composition with a partial identity:

2.1 Proposition. $\overline{(\mathbb{I} \cap \Delta);\mathbb{T}} = (\mathbb{I} \cap \overline{\Delta});\mathbb{T}$ for an arbitrary homogeneous relation Δ.

Proof: $\mathbb{T} = \mathbb{I};\mathbb{T} = \big[\mathbb{I} \cap (\Delta \cup \overline{\Delta})\big];\mathbb{T} = \big[(\mathbb{I} \cap \Delta) \cup (\mathbb{I} \cap \overline{\Delta})\big];\mathbb{T} = (\mathbb{I} \cap \Delta);\mathbb{T} \cup (\mathbb{I} \cap \overline{\Delta});\mathbb{T}$ implies $\overline{(\mathbb{I} \cap \Delta);\mathbb{T}} \subseteq (\mathbb{I} \cap \overline{\Delta});\mathbb{T}$, thus proving direction "$\subseteq$".

For "$\supseteq$", we use that $\mathbb{I} \cap \overline{\Delta} \subseteq \mathbb{I}$ is univalent, prior to applying the Schröder rule:
$(\mathbb{I} \cap \overline{\Delta})^\top;(\mathbb{I} \cap \Delta);\mathbb{T} = [(\mathbb{I} \cap \overline{\Delta});\mathbb{I} \cap (\mathbb{I} \cap \overline{\Delta});\Delta];\mathbb{T} \subseteq [\mathbb{I} \cap \overline{\Delta} \cap \Delta];\mathbb{T} = \perp\!\!\!\perp$ □

It is relatively hard to see: this specializes Prop. 5.6 in [Sch11] for a homogeneous relation Δ:
$\overline{(\mathbb{I} \cap \Delta);R} = (\mathbb{I} \cap \overline{\Delta});\mathbb{T} \cup (\mathbb{I} \cap \Delta);\overline{R}$

Another rule that sometimes proves helpful is the following:

2.2 Proposition. For any two mappings $f, g : X \longrightarrow Y$, this rule holds:
$(\overline{f} \cap g);\mathbb{T} = (f \cap \overline{g});\mathbb{T}$

Proof: $(\overline{f} \cap g);\mathbb{T} = (f;\overline{\mathbb{I}} \cap g);\mathbb{T} \subseteq (f \cap g;\overline{\mathbb{I}});(\overline{\mathbb{I}} \cap f^\top;g);\mathbb{T} = (f \cap \overline{g});(\overline{\mathbb{I}} \cap f^\top;g);\mathbb{T} \subseteq (f \cap \overline{g});\mathbb{T}$ □

There exist two resp. three versions of an interpretation. The first one takes two mappings f, g which never assign the same value. In this case both sides result in $\mathbb{T}$. Then there may be two mappings with one or more values identical. In this case, precisely the respective arguments lead to $\mathbf{0}$-rows; they may even lead to $\perp\!\!\!\perp$ when $f = g$.

3 Symmetric quotient and membership

When a non-commutative composition is available, one usually looks for the left and the right residual, defined via

$$A;B \subseteq C \iff A \subseteq \overline{\overline{C};B^\top} =: C/B \quad \text{and} \quad A;B \subseteq C \iff B \subseteq \overline{A^\top;\overline{C}} =: A\backslash C.$$

Residuations have been studied intensively, not least in the context of Heyting algebras. We prove some rules for residuals:

3.1 Proposition (*Residue cancellation*)**.** The following formulae hold for arbitrary relations Q, R, T — provided typing is correct:

i) $(Q\backslash R)/T = Q\backslash(R/T)$

ii) $Q\backslash Q = (Q\backslash Q)/(Q\backslash Q)$

iii) $Q/(R;U) \subseteq (Q;U^\top)/R$ if U is total

Proof: i) $(Q\backslash R)/T = \overline{\overline{\overline{Q^\top;\overline{R}}};T^\top} = \overline{Q^\top;\overline{R};T^\top}$ and symmetrically to the other side.

ii) $Q\backslash Q = \overline{Q^\top;\overline{Q}}$ implies that $(Q\backslash Q)/(Q\backslash Q) = \overline{\overline{\overline{Q^\top;\overline{Q}}};\overline{Q^\top;\overline{Q}}^\top} = \overline{Q^\top;\overline{Q};\overline{\overline{Q}^\top;Q}} = \overline{Q^\top;\overline{Q}}$, since $\overline{Q};\overline{\overline{Q}^\top;Q} = \overline{Q}$, which in turn follows from Schröder's rule and reflexivity of $\overline{\overline{Q}^\top;Q}$.

iii) $\mathbb{T} = \mathbb{T};U^\top = (Q \cup \overline{Q});U^\top = Q;U^\top \cup \overline{Q};U^\top \iff \overline{Q;U^\top} \subseteq \overline{Q};U^\top$
$\implies \overline{Q;U^\top};R^\top \subseteq \overline{Q};U^\top;R^\top \iff \overline{\overline{Q};U^\top;R^\top} \subseteq \overline{\overline{Q;U^\top};R^\top}$ □

Intersecting such residuals in $\mathtt{syq}(R,S) := \overline{R^{\mathsf{T}};\overline{S}} \cap \overline{\overline{R}^{\mathsf{T}};S}$, the *symmetric quotient* $\mathtt{syq}(R,S) : W \longrightarrow Z$ of two relations $R : V \longrightarrow W$ and $S : V \longrightarrow Z$ is defined. Symmetric quotients serve the purpose of 'column comparison':

$$\big[\mathtt{syq}(R,S)\big]_{wz} = \forall v \in V : R_{vw} \longleftrightarrow S_{vz}.$$

The following result may easily be understood. If a column of A and the corresponding one of B are equal to some column of C, then also their intersection and union will be equal.

3.2 Proposition. For arbitrary relations A, B, C with all the same source always

$$\mathtt{syq}(A,C) \cap \mathtt{syq}(B,C) \subseteq \mathtt{syq}(A \cap B, C) \cap \mathtt{syq}(A \cup B, C).$$

Proof: For inclusion in the first term, we expand the symmetric quotients and negate to obtain

$$\overline{A \cap B}^{\mathsf{T}};C \cup (A \cap B)^{\mathsf{T}};\overline{C} \subseteq \overline{A}^{\mathsf{T}};C \cup A^{\mathsf{T}};\overline{C} \;\cup\; \overline{B}^{\mathsf{T}};C \cup B^{\mathsf{T}};\overline{C},$$

which is obviously satisfied. This is then used to prove the other part.

$$\mathtt{syq}(A \cup B, C) = \mathtt{syq}(\overline{A \cup B}, \overline{C}) = \mathtt{syq}(\overline{A} \cap \overline{B}, \overline{C}) \quad \text{now applying the former}$$
$$\supseteq \mathtt{syq}(\overline{A}, \overline{C}) \cap \mathtt{syq}(\overline{B}, \overline{C}) = \mathtt{syq}(A,C) \cap \mathtt{syq}(B,C) \qquad \square$$

The illustration of the symmetric quotient is as follows:

$$R = \begin{array}{r} \\ \text{US} \\ \text{French} \\ \text{German} \\ \text{British} \\ \text{Spanish} \end{array}
\begin{array}{c}
\begin{array}{ccccccccccccc} \text{A} & \text{K} & \text{Q} & \text{J} & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 \end{array} \\
\left(\begin{array}{ccccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&1&0&0&0&0&0&1&0&0&0&0&0\\
0&0&1&0&0&0&1&1&0&1&0&1&0\\
0&1&1&0&0&0&0&1&0&0&0&0&1\\
0&0&0&1&0&0&1&0&0&1&1&0&1
\end{array}\right)
\end{array}$$

$$S = \begin{array}{r} \\ \text{US} \\ \text{French} \\ \text{German} \\ \text{British} \\ \text{Spanish} \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} \text{Jan} & \text{Feb} & \text{Mar} & \text{Apr} & \text{May} & \text{Jun} & \text{Jul} & \text{Aug} & \text{Sep} & \text{Oct} & \text{Nov} & \text{Dec} \end{array} \\
\left(\begin{array}{cccccccccccc}
0&0&0&1&0&1&1&1&0&1&0&0\\
1&0&0&1&0&0&1&0&0&1&0&0\\
1&1&0&0&1&1&0&1&0&0&0&1\\
1&1&0&0&0&0&1&0&1&0&1&1\\
0&0&0&1&0&1&1&1&0&0&0&0
\end{array}\right)
\end{array}$$

$$\begin{array}{r} \\ \text{A} \\ \text{K} \\ \text{Q} \\ \text{J} \\ 10 \\ 9 \\ 8 \\ 7 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} \text{Jan} & \text{Feb} & \text{Mar} & \text{Apr} & \text{May} & \text{Jun} & \text{Jul} & \text{Aug} & \text{Sep} & \text{Oct} & \text{Nov} & \text{Dec} \end{array} \\
\left(\begin{array}{cccccccccccc}
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right)
\end{array}$$

Fig. 3.1 R, S and $\mathtt{syq}(R,S)$

The symmetric quotient shows which columns of the left are equal to columns of the right relation in $\mathtt{syq}(R,S)$, with S conceived as the denominator.

It is extremely helpful that the symmetric quotient enjoys certain cancellation properties. These are far from being broadly known. Just minor side conditions have to be observed. In any of the following propositions correct typing is assumed. What is more important is that one may calculate with the symmetric quotient in a fairly traditional algebraic way. Proofs may be found in [Sch11].

3.3 Theorem. Arbitrary relations A, B satisfy in analogy to $a \cdot \frac{b}{a} = b$:

i) $A;\mathtt{syq}(A,B) = B \cap \mathbb{T};\mathtt{syq}(A,B)$,

ii) $\mathtt{syq}(A,B)$ surjective $\implies$ $A;\mathtt{syq}(A,B) = B$. $\square$

3.4 Theorem. Arbitrary relations A, B, C satisfy in analogy to $\frac{b}{a} \cdot \frac{c}{b} = \frac{c}{a}$

i) $\mathtt{syq}(A,B);\mathtt{syq}(B,C) = \mathtt{syq}(A,C) \cap \mathtt{syq}(A,B);\mathbb{T}$

$$= \mathtt{syq}(A,C) \cap \mathbb{T};\mathtt{syq}(B,C)$$

ii) If $\mathtt{syq}(A,B)$ is total, or if $\mathtt{syq}(B,C)$ is surjective, then

$\mathtt{syq}(A,B);\mathtt{syq}(B,C) = \mathtt{syq}(A,C)$. □

3.5 Theorem. Assuming arbitrary relations X, Y, Z, always in analogy to $\frac{z}{x} \,:\, \frac{y}{x} = \frac{z}{y}$

- $\mathtt{syq}(X,Y) \setminus \mathtt{syq}(Z,X) \supseteq \mathtt{syq}(Z,Y)$
- $\mathtt{syq}(\mathtt{syq}(X,Y), \mathtt{syq}(X,Z)) \supseteq \mathtt{syq}(Y,Z)$
- $\mathtt{syq}(\mathtt{syq}(X,Y), \mathtt{syq}(X,Z)) = \mathtt{syq}(Y,Z)$ if $\mathtt{syq}(X,Y)$ and $\mathtt{syq}(X,Z)$ are surjective

□

Here is another basic rule:

3.6 Proposition. For a surjective mapping f always $\mathtt{syq}(X, f;Y) \subseteq \mathtt{syq}(f^\top;X, Y)$.

Proof: $\iff \quad \overline{\overline{X}^\top;f;Y} \cap \overline{X^\top;\overline{f;Y}} \subseteq \overline{\overline{X^\top;f};Y} \cap \overline{X^\top;f;\overline{Y}}$

Above, the second terms are equal since f is a mapping. Containment of the first ones:

$$\iff \quad \overline{X^\top;f};Y \subseteq \overline{X}^\top;f;Y \quad \Longleftarrow \quad \overline{X^\top;f} \subseteq \overline{X}^\top;f \quad \iff \quad \mathbb{T} = X^\top;f \cup \overline{X}^\top;f = \mathbb{T};f$$ □

4 Membership and singleton injection

The symmetric quotient is used to introduce *membership relations* $\varepsilon : V \longrightarrow \mathcal{P}(V)$ between a set V and its powerset $\mathcal{P}(V)$ or $\mathbf{2}^V$. These can be characterized algebraically up to isomorphism demanding $\mathtt{syq}(\varepsilon,\varepsilon) \subseteq \mathbb{I}$ and surjectivity of $\mathtt{syq}(\varepsilon, R)$ for all R. With a membership ε, the powerset ordering is easily described as $\Omega = \overline{\varepsilon^\top;\overline{\varepsilon}}$. Also least upper bounds with regard to Ω may be expressed via membership and symmetric quotient, making this a very powerful tool; see [Sch11].

4.1 Proposition. If ε is the membership relation and Ω the corresponding powerset ordering, the following equations hold for arbitrary relations X:

i) $\varepsilon;\overline{\varepsilon^\top;X} = \overline{X}$ and $\overline{\varepsilon};\overline{\overline{\varepsilon}^\top;X} = \overline{X}$,

ii) $\mathtt{lub}_\Omega(X) = \mathtt{syq}(\varepsilon, \varepsilon;X)$. □

We also introduce **singleton injection** $\sigma := \mathtt{syq}(\mathbb{I},\varepsilon)$ and **atoms** $a := \sigma^\mathsf{T};\sigma$.

$$\varepsilon = \begin{array}{c} \\ a\\ b\\ c\\ d \end{array}\!\!\begin{array}{c} \begin{array}{cccccccccccccccc} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & abc & \{d\} & \{a,d\} & \{b,d\} & abd & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \end{array} \\ \begin{pmatrix} 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\ 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1\\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1\\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \end{pmatrix} \end{array}$$

$$\sigma = \begin{array}{c} \\ a\\ b\\ c\\ d \end{array}\!\!\begin{array}{c} \begin{array}{cccccccccccccccc} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & abc & \{d\} & \{a,d\} & \{b,d\} & abd & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \end{array} \\ \begin{pmatrix} 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \end{pmatrix} \end{array}$$

$$\mathbb{T};\sigma^\mathsf{T};\sigma = \begin{pmatrix} 0&1&1&0&1&0&0&0&1&0&0&0&0&0&0&0 \end{pmatrix}$$

$$\begin{array}{r} \\ \{\}\\ \{a\}\\ \{b\}\\ \{a,b\}\\ \{c\}\\ \{a,c\}\\ \{b,c\}\\ \{a,b,c\}\\ \{d\}\\ \{a,d\}\\ \{b,d\}\\ abd\\ \{c,d\}\\ \{a,c,d\}\\ \{b,c,d\}\\ \{a,b,c,d\} \end{array}\!\!\begin{array}{c} \begin{array}{cccccccccccccccc} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & abd & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \end{array} \\ \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \end{array}$$

Fig. 4.1 ε, singleton injection $\sigma := \mathtt{syq}(\mathbb{I},\varepsilon)$ and atoms as vector $\sigma^\mathsf{T};\sigma;\mathbb{T}$ as well as diagonal $\sigma^\mathsf{T};\sigma$

The following results correspond to the lowest level of element-is-contained-in-set considerations. They are fairly intuitive and easy to understand from Fig. 4.1. The basic purpose of these statements is to make these tiny set arguments work together with more advanced algebraic mechanisms.

4.2 Lemma. i) $\sigma;\varepsilon^\mathsf{T} = \mathbb{I}$

ii) $\overline{\overline{\mathbb{I}};\varepsilon} = \sigma \cup \overline{\mathbb{T};\varepsilon}$

iii) $\sigma;\Omega = \varepsilon \qquad \sigma;\Omega^\mathsf{T} = \sigma \cup \overline{\mathbb{T};\varepsilon}$

iv) $\varepsilon = \sigma \cup (\varepsilon \cap \overline{\mathbb{T};\sigma})$

v) $\overline{\mathbb{I}};\varepsilon \cap \overline{\overline{\mathbb{I}};\sigma} = \mathbb{T};\varepsilon \cap \overline{\mathbb{T};\sigma}$

vi) $\Omega \cap \varepsilon^\mathsf{T};\varepsilon = \Omega \cap \varepsilon^\mathsf{T};\mathbb{T}$

vii) $(\Omega \cap \varepsilon^\mathsf{T};\varepsilon);\varepsilon^\mathsf{T} = \varepsilon^\mathsf{T};\mathbb{T}$

viii) $(\Omega \cap \varepsilon^\mathsf{T};\mathbb{T})^\mathsf{T};(\Omega \cap \varepsilon^\mathsf{T};\mathbb{T}) = \varepsilon^\mathsf{T};\varepsilon$

Proof: i) $\sigma;\varepsilon^\mathsf{T} = \left[\varepsilon;\sigma^\mathsf{T}\right]^\mathsf{T} = \left[\varepsilon;\mathtt{syq}(\varepsilon,\mathbb{I})\right]^\mathsf{T} = \mathbb{I}$

ii) $\sigma = \mathtt{syq}(\mathbb{I},\varepsilon) = \overline{\overline{\mathbb{I}};\varepsilon} \cap \varepsilon$ by definition and $\overline{\overline{\mathbb{I}};\varepsilon} \supseteq \overline{\mathbb{T};\varepsilon}$ result in "$\supseteq$".

"$\subseteq$" means $\overline{\overline{\mathbb{I}};\varepsilon} \subseteq \sigma \cup \overline{\mathbb{T};\varepsilon} = \left[\overline{\overline{\mathbb{I}};\varepsilon} \cap \varepsilon\right] \cup \overline{\mathbb{T};\varepsilon}$

$\iff \quad \mathbb{T} = \overline{\mathbb{I}};\varepsilon \cup \left[\overline{\overline{\mathbb{I}};\varepsilon} \cap \varepsilon\right] \cup \overline{\mathbb{T};\varepsilon} = \left[\overline{\mathbb{I}};\varepsilon \cup \overline{\overline{\mathbb{I}};\varepsilon} \cup \overline{\mathbb{T};\varepsilon}\right] \cap \left[\overline{\mathbb{I}};\varepsilon \cup \varepsilon \cup \overline{\mathbb{T};\varepsilon}\right]$, which is true.

iii) $\sigma;\Omega = \sigma;\overline{\varepsilon^\mathsf{T};\overline{\varepsilon}} = \overline{\sigma;\varepsilon^\mathsf{T};\overline{\varepsilon}} = \overline{\mathbb{I};\overline{\varepsilon}} = \varepsilon, \qquad$ using (i)

$\sigma;\Omega^{\mathsf{T}} = \sigma;\overline{\overline{\varepsilon^{\mathsf{T}}};\varepsilon} = \overline{\overline{\sigma;\varepsilon^{\mathsf{T}}};\varepsilon} = \overline{\overline{\mathbb{I}};\varepsilon} = \sigma \cup \overline{\mathbb{T};\varepsilon},$ using (i,ii)

iv) "$\supseteq$" is obvious. For "$\subseteq$", it suffices to prove $\mathbb{T};\sigma \cap \varepsilon \subseteq (\mathbb{T} \cap \varepsilon;\sigma^{\mathsf{T}});(\sigma \cap \mathbb{T};\varepsilon) \subseteq \sigma$ using (i).

v) $\mathbb{T};\varepsilon \cap \overline{\mathbb{T};\sigma} = (\mathbb{I};\varepsilon \cup \overline{\mathbb{I}};\varepsilon) \cap \overline{\mathbb{I};\sigma \cup \overline{\mathbb{I}};\sigma} = (\mathbb{I};\varepsilon \cup \overline{\mathbb{I}};\varepsilon) \cap \overline{\sigma} \cap \overline{\overline{\mathbb{I}};\sigma} = (\mathbb{I};\varepsilon \cup \overline{\mathbb{I}};\varepsilon) \cap (\overline{\mathbb{I}};\varepsilon \cup \overline{\varepsilon}) \cap \overline{\overline{\mathbb{I}};\sigma}$
$= \left[\overline{\mathbb{I}};\varepsilon \cup (\varepsilon \cap \overline{\varepsilon})\right] \cap \overline{\overline{\mathbb{I}};\sigma} = \overline{\mathbb{I}};\varepsilon \cap \overline{\overline{\mathbb{I}};\sigma}$

vi) This follows with the Dedekind rule from
$\varepsilon^{\mathsf{T}};\mathbb{T} \cap \Omega \subseteq (\varepsilon^{\mathsf{T}} \cap \Omega;\mathbb{T});(\mathbb{T} \cap \varepsilon;\Omega) \subseteq \varepsilon^{\mathsf{T}};\varepsilon;\Omega = \varepsilon^{\mathsf{T}};\varepsilon.$

vii) We start with formally showing the intuitively clear fact $\Omega;\varepsilon^{\mathsf{T}} = \mathbb{T}$:
$\Omega;\varepsilon^{\mathsf{T}} \supseteq (\Omega \cap \overline{\mathbb{T};\overline{\varepsilon}});(\varepsilon^{\mathsf{T}} \cap \overline{\overline{\varepsilon^{\mathsf{T}}};\mathbb{T}}) = \overline{\mathbb{T};\overline{\varepsilon}};\overline{\overline{\varepsilon^{\mathsf{T}}};\mathbb{T}}$ trivial
$\supseteq \mathtt{syq}(\mathbb{T},\varepsilon);\mathtt{syq}(\varepsilon,\mathbb{T}) = \mathtt{syq}(\mathbb{T},\mathbb{T}) = \mathbb{T}$ [Sch11] Prop. 8.13.ii; $\mathtt{syq}(\varepsilon,\mathbb{T})$ is surjective
which is used together with (vi) in the following chain of reasoning
$(\Omega \cap \varepsilon^{\mathsf{T}};\varepsilon);\varepsilon^{\mathsf{T}} = (\Omega \cap \varepsilon^{\mathsf{T}};\mathbb{T});\varepsilon^{\mathsf{T}} = \Omega;\varepsilon^{\mathsf{T}} \cap \varepsilon^{\mathsf{T}};\mathbb{T} = \mathbb{T} \cap \varepsilon^{\mathsf{T}};\mathbb{T} = \varepsilon^{\mathsf{T}};\mathbb{T}.$

viii) We recall the definition of singleton injection $\sigma := \mathtt{syq}(\mathbb{I},\varepsilon)$ and use (i,iii):
$\varepsilon = \varepsilon \cap \mathbb{T} = \varepsilon \cap \mathbb{I};\mathbb{T} = \sigma;\Omega \cap \sigma;\varepsilon^{\mathsf{T}};\mathbb{T} = \sigma;(\Omega \cap \varepsilon^{\mathsf{T}};\mathbb{T})$ since σ is univalent
Therefore
$\varepsilon^{\mathsf{T}};\varepsilon = (\Omega \cap \varepsilon^{\mathsf{T}};\mathbb{T})^{\mathsf{T}};\sigma^{\mathsf{T}};\sigma;(\Omega \cap \varepsilon^{\mathsf{T}};\mathbb{T}) \subseteq (\Omega \cap \varepsilon^{\mathsf{T}};\mathbb{T})^{\mathsf{T}};(\Omega \cap \varepsilon^{\mathsf{T}};\mathbb{T})$ since σ is univalent
The other inclusion "$\subseteq$" follows with (vi) from
$(\Omega^{\mathsf{T}} \cap \mathbb{T};\varepsilon);(\Omega \cap \varepsilon^{\mathsf{T}};\mathbb{T}) = (\Omega^{\mathsf{T}} \cap \varepsilon^{\mathsf{T}};\varepsilon);(\Omega \cap \varepsilon^{\mathsf{T}};\varepsilon) \subseteq \varepsilon^{\mathsf{T}};\varepsilon;\Omega = \varepsilon^{\mathsf{T}};\varepsilon$ □

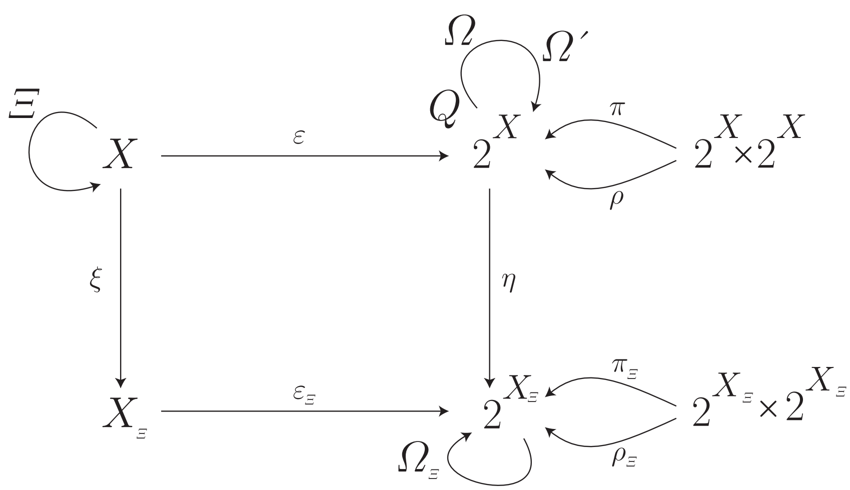

Fig. 4.2 Quotient of membership

4.3 Proposition. Let be given the membership relation $\varepsilon : X \longrightarrow \mathbf{2}^X$ and an equivalence relation $\Xi : X \longrightarrow X$ and its natural projection $\xi : X \longrightarrow X_\Xi$ satisfying $\xi;\xi^{\mathsf{T}} = \Xi$. Then

i) $\Omega' := \overline{\varepsilon^{\mathsf{T}};\overline{\Xi;\varepsilon}}$ is a preorder.

ii) $Q := \mathtt{syq}(\Xi;\varepsilon, \Xi;\varepsilon)$ is an equivalence satisfying $Q = \Omega' \cap \Omega'^{\mathsf{T}}$.

Let $\eta : \mathbf{2}^X \longrightarrow \mathbf{2}^{X_\Xi}$ denote its natural projection, i.e., the mapping that satisfies $Q = \eta;\eta^{\mathsf{T}}$.

iii) $\varepsilon;Q = \Xi;\varepsilon$

iv) $\varepsilon_\Xi := \xi^\mathsf{T};\varepsilon;\eta$ satisfies the properties of a membership relation.

v) $\xi^\mathsf{T};\varepsilon = \varepsilon_\Xi;\eta^\mathsf{T} \qquad \Omega';\eta = \eta;\Omega_\Xi$

Proof: We recall that for an equivalence Ψ, in general $\Psi;\overline{\Psi;Y} = \overline{\Psi;Y}$ and $\overline{Z;\Psi};\Psi = \overline{Z;\Psi}$, and that its natural projection is a surjective mapping.

i) Reflexivity holds since $\mathbb{I} \subseteq \Omega = \overline{\varepsilon^\mathsf{T};\overline{\varepsilon}} \subseteq \overline{\varepsilon^\mathsf{T};\overline{\Xi;\varepsilon}}$, while transitivity follows from

$$\overline{\varepsilon^\mathsf{T};\overline{\Xi}};\Xi;\varepsilon = \overline{\varepsilon^\mathsf{T};\overline{\Xi}};\varepsilon \subseteq \overline{\varepsilon^\mathsf{T};\overline{\Xi}};\varepsilon \quad\Longleftrightarrow\quad \overline{\Xi;\varepsilon};\overline{\overline{\varepsilon^\mathsf{T};\overline{\Xi}};\varepsilon} \subseteq \overline{\Xi;\varepsilon} \quad\Longrightarrow\quad \varepsilon^\mathsf{T};\overline{\Xi;\varepsilon};\overline{\overline{\varepsilon^\mathsf{T};\overline{\Xi}};\varepsilon} \subseteq \varepsilon^\mathsf{T};\overline{\Xi;\varepsilon}$$

$$\Longleftrightarrow \quad \overline{\varepsilon^\mathsf{T};\overline{\Xi;\varepsilon}};\overline{\varepsilon^\mathsf{T};\overline{\Xi;\varepsilon}} \subseteq \overline{\varepsilon^\mathsf{T};\overline{\Xi;\varepsilon}}, \quad \text{i.e. } \Omega';\Omega' \subseteq \Omega'$$

ii) A relation $\mathtt{syq}(A,A)$ is always an equivalence following [Sch11, Prop. 8.14.i]; furthermore

$$\Omega' \cap \Omega'^\mathsf{T} = \overline{\varepsilon^\mathsf{T};\overline{\overline{\Xi};\varepsilon}} \cap \overline{\overline{\overline{\Xi};\varepsilon}^\mathsf{T};\varepsilon} = \overline{\varepsilon^\mathsf{T};\Xi;\overline{\overline{\Xi};\varepsilon}} \cap \overline{\overline{\overline{\Xi};\varepsilon}^\mathsf{T};\Xi;\varepsilon} = \mathtt{syq}(\Xi;\varepsilon, \Xi;\varepsilon) = Q$$

iii) $\Xi;\varepsilon = \varepsilon;\mathtt{syq}(\varepsilon, \Xi;\varepsilon)$ since ε is a membership
$\subseteq \varepsilon;\mathtt{syq}(\Xi;\varepsilon, \Xi;\Xi;\varepsilon) = \varepsilon;\mathtt{syq}(\Xi;\varepsilon, \Xi;\varepsilon) = \varepsilon;Q$ [Sch11, Prop. 8.16.i]
$\subseteq \Xi;\varepsilon;\mathtt{syq}(\Xi;\varepsilon, \Xi;\varepsilon) = \Xi;\varepsilon$ since always $A;\mathtt{syq}(A,A) = A$

iv) $\mathtt{syq}(\varepsilon_\Xi, \varepsilon_\Xi) = \mathtt{syq}(\xi^\mathsf{T};\varepsilon;\eta, \xi^\mathsf{T};\varepsilon;\eta) = \eta^\mathsf{T};\mathtt{syq}(\xi^\mathsf{T};\varepsilon, \xi^\mathsf{T};\varepsilon);\eta$ due to [Sch11, Prop. 8.18].
$= \eta^\mathsf{T};\mathtt{syq}(\xi;\xi^\mathsf{T};\varepsilon, \xi;\xi^\mathsf{T};\varepsilon);\eta$ due to [Sch11, Prop. 8.16.i] since ξ is a surjective mapping
$= \eta^\mathsf{T};\mathtt{syq}(\Xi;\varepsilon, \Xi;\varepsilon);\eta = \eta^\mathsf{T};Q;\eta = \eta^\mathsf{T};\eta;\eta^\mathsf{T};\eta = \mathbb{I};\mathbb{I} = \mathbb{I}$

Assuming an arbitrary X that is acceptable with regard to typing,
$\mathbb{T};\mathtt{syq}(\varepsilon_\Xi, X) = \mathbb{T};\mathtt{syq}(\xi^\mathsf{T};\varepsilon;\eta, X)$
$= \mathbb{T};\eta^\mathsf{T};\mathtt{syq}(\xi^\mathsf{T};\varepsilon, X)$ due to [Sch11, Prop. 8.18] since $\xi^\mathsf{T};\varepsilon = \xi^\mathsf{T};\varepsilon;Q$, see above
$= \mathbb{T};\mathtt{syq}(\xi^\mathsf{T};\varepsilon, X)$
$\supseteq \mathbb{T};\mathtt{syq}(\varepsilon, \xi;X)$ Prop. 3.6
$= \mathbb{T}$ because ε is a membership.

v) $\xi^\mathsf{T};\varepsilon = \xi^\mathsf{T};\xi;\xi^\mathsf{T};\varepsilon = \xi^\mathsf{T};\Xi;\varepsilon = \xi^\mathsf{T};\varepsilon;Q = \xi^\mathsf{T};\varepsilon;\eta;\eta^\mathsf{T} = \varepsilon_\Xi;\eta^\mathsf{T}$

$$\Omega';\eta = \overline{\varepsilon^\mathsf{T};\overline{\Xi;\varepsilon}};\eta = \overline{\varepsilon^\mathsf{T};\overline{\Xi;\Xi;\varepsilon}};\eta = \overline{\varepsilon^\mathsf{T};\overline{\Xi;\varepsilon;Q}};\eta = \overline{\varepsilon^\mathsf{T};\overline{\Xi;\varepsilon;\eta;\eta^\mathsf{T}}};\eta = \overline{\varepsilon^\mathsf{T};\overline{\Xi;\varepsilon;\eta}};\eta^\mathsf{T};\eta = \overline{\varepsilon^\mathsf{T};\overline{\Xi;\varepsilon;\eta}}$$

$$= \overline{\varepsilon^\mathsf{T};\Xi;\overline{\Xi;\varepsilon;\eta}} = \overline{Q;\varepsilon^\mathsf{T};\overline{\Xi;\varepsilon;\eta}} = \overline{\eta;\eta^\mathsf{T};\varepsilon^\mathsf{T};\overline{\xi;\xi^\mathsf{T};\varepsilon;\eta}} = \eta;\overline{\eta^\mathsf{T};\varepsilon^\mathsf{T};\xi;\overline{\xi^\mathsf{T};\varepsilon;\eta}} = \eta;\overline{\varepsilon_\Xi^\mathsf{T};\overline{\varepsilon_\Xi}} = \eta;\Omega_\Xi \qquad \square$$

5 Power operations

There is an interesting interrelationship from relations to their counterparts between the corresponding powersets. It offers the possibility to work algebraically at situations where this has so far not been the classical approach; some has already been collected in [Sch11].

5.1 Definition. Let any relation $R : X \longrightarrow Y$ be given together with membership relations $\varepsilon : X \longrightarrow \mathbf{2}^X, \varepsilon' : Y \longrightarrow \mathbf{2}^Y$. Then the corresponding **existential image mapping** is defined as $\vartheta_R := \mathtt{syq}(R^{\mathsf{T}};\varepsilon, \varepsilon')$. One may correspondingly study the **inverse image mapping** defined as $\vartheta_{R^{\mathsf{T}}} = \mathtt{syq}(R;\varepsilon', \varepsilon)$. □

We recall an interesting fact concerning the existential image; see [Sch11]. Referring to [dRE98], the pair $\varepsilon, \varepsilon'$ constitutes an L-simulation of ϑ_R by R, and in addition, $\varepsilon^{\mathsf{T}}, \varepsilon'^{\mathsf{T}}$ show an L^{T}-simulation of R by ϑ_R. In total, we have for an existential image the equality

$$\varepsilon^{\mathsf{T}};R = \vartheta_R;\varepsilon'^{\mathsf{T}}.$$

Correspondingly, an application of this simulation rule to R^{T} instead of R reads

$$\varepsilon'^{\mathsf{T}};R^{\mathsf{T}} = \vartheta_{R^{\mathsf{T}}};\varepsilon^{\mathsf{T}}, \qquad \text{or else} \qquad R;\varepsilon' = \varepsilon;\vartheta^{\mathsf{T}}_{R^{\mathsf{T}}}.$$

5.2 Proposition. The existential image and the inverse image also satisfy formulae with respect to the powerset orderings:

i) $\Omega';\vartheta_{f^{\mathsf{T}}} \subseteq \vartheta_{f^{\mathsf{T}}};\Omega$ if f is a mapping,

ii) $\Omega;\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}} = \vartheta_f;\Omega'$ if f is a mapping.

Proof: i) Via shunting the claim is $\Omega' \subseteq \vartheta_{f^{\mathsf{T}}};\Omega;\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}}$, which we prove in negated form:

$$\varepsilon'^{\mathsf{T}};\overline{\varepsilon'} \supseteq \varepsilon'^{\mathsf{T}};f^{\mathsf{T}};f;\overline{\varepsilon'} = \varepsilon'^{\mathsf{T}};f^{\mathsf{T}};\overline{f;\varepsilon'} = \varepsilon'^{\mathsf{T}};f^{\mathsf{T}};\overline{\varepsilon;\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}}} = \vartheta_{f^{\mathsf{T}}};\varepsilon^{\mathsf{T}};\overline{\varepsilon};\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}} = \vartheta_{f^{\mathsf{T}}};\overline{\Omega};\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}} = \overline{\vartheta_{f^{\mathsf{T}}};\Omega;\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}}}$$

ii) $\Omega;\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}} = \overline{\varepsilon^{\mathsf{T}};\overline{\varepsilon}};\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}} = \overline{\varepsilon^{\mathsf{T}};\overline{\varepsilon;\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}}}} = \overline{\varepsilon^{\mathsf{T}};\overline{f;\varepsilon'}} = \overline{\varepsilon^{\mathsf{T}};f;\overline{\varepsilon'}} = \overline{\vartheta_f;\varepsilon'^{\mathsf{T}};\overline{\varepsilon'}} = \vartheta_f;\overline{\varepsilon'^{\mathsf{T}};\overline{\varepsilon'}} = \vartheta_f;\Omega'$ □

Another rule combines the inverse image with the singleton injection.

5.3 Proposition. i) Any relation $R : X \longrightarrow Y$ with σ_X, σ_Y the singleton injections satisfies

$$\sigma_X;\vartheta^{\mathsf{T}}_{R^{\mathsf{T}}};\sigma^{\mathsf{T}}_Y \subseteq R \quad \text{and} \quad \varepsilon_X;\vartheta^{\mathsf{T}}_{R^{\mathsf{T}}};\sigma^{\mathsf{T}}_Y = R.$$

ii) When f is a mapping, this sharpens to $\sigma_X;\vartheta_f = f;\sigma_Y$.

Proof: i) $\sigma_X;\vartheta^{\mathsf{T}}_{R^{\mathsf{T}}};\sigma^{\mathsf{T}}_Y \subseteq \varepsilon_X;\vartheta^{\mathsf{T}}_{R^{\mathsf{T}}};\sigma^{\mathsf{T}}_Y = R;\varepsilon_Y;\sigma^{\mathsf{T}}_Y = R$ Prop. 5.2.i

ii) $\sigma_X;\vartheta_f = \sigma_X;\mathtt{syq}(f^{\mathsf{T}};\varepsilon_X, \varepsilon_Y)$ definition of symmetric quotient
$= \mathtt{syq}(f^{\mathsf{T}};\varepsilon_X;\sigma^{\mathsf{T}}_X, \varepsilon_Y)$ Prop. 8.16.ii of [Sch11]
$= \mathtt{syq}(f^{\mathsf{T}}, \varepsilon_Y) = \mathtt{syq}(\mathbb{I};f^{\mathsf{T}}, \varepsilon_Y) = f;\mathtt{syq}(\mathbb{I}, \varepsilon_Y) = f;\sigma_Y$ Prop. 5.2.i □

The following rules are not unimportant when, in a forthcoming paper, continuity is studied in topology and transferred to a point-free relation-algebraic version.

$$\begin{tikzcd}
2^X \arrow[loop left, "\Omega"] \arrow[rr, shift left, "\vartheta_R"] \arrow[rr, bend right, "\zeta"'] & & 2^{X'} \arrow[ll, shift left, "\vartheta_{R^\mathsf{T}}"] \arrow[loop right, "\Omega'"] \\
X \arrow[u, "\varepsilon"] \arrow[rr, "R"'] & & X' \arrow[u, "\varepsilon'"']
\end{tikzcd}$$

$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{matrix} \mathrm{a} & \mathrm{b} & \mathrm{c} & \mathrm{d} \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

$$\vartheta_R = \begin{array}{r|cccccccccccccccc}
 & \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \\
\hline
\{\} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{1\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\{2\} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{1,2\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\{3\} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{1,3\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\{2,3\} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{1,2,3\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\{4\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{1,4\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\{2,4\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{1,2,4\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\{3,4\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\{1,3,4\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\{2,3,4\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\{1,2,3,4\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\{5\} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{1,5\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\{2,5\} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{1,2,5\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\{3,5\} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{1,3,5\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\{2,3,5\} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{1,2,3,5\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\{4,5\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\{1,4,5\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\{2,4,5\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\{1,2,4,5\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\{3,4,5\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\{1,3,4,5\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\{2,3,4,5\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\{1,2,3,4,5\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}$$

$$\vartheta_{R^\mathsf{T}} = \begin{array}{r|cccccccccccccccccccccccccccccccc}
 & \{\} & \{1\} & \{2\} & \{1,2\} & \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} & \{4\} & \{1,4\} & \{2,4\} & \{1,2,4\} & \{3,4\} & \{1,3,4\} & \{2,3,4\} & \{1,2,3,4\} & \{5\} & \{1,5\} & \{2,5\} & \{1,2,5\} & \{3,5\} & \{1,3,5\} & \{2,3,5\} & \{1,2,3,5\} & \{4,5\} & \{1,4,5\} & \{2,4,5\} & \{1,2,4,5\} & \{3,4,5\} & \{1,3,4,5\} & \{2,3,4,5\} & \{1,2,3,4,5\} \\
\hline
\{\} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{a\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{b\} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{a,b\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{c\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{a,c\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{b,c\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{a,b,c\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{d\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{a,d\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\{b,d\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{a,b,d\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\{c,d\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\{a,c,d\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\{b,c,d\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\{a,b,c,d\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}$$

Fig. 5.1 Existential and inverse images

5.4 Proposition. Let $f : X \longrightarrow Y$ be an arbitrary mapping. Then

i) $\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}};\vartheta_{f^{\mathsf{T}}};\vartheta_f = \vartheta^{\mathsf{T}}_{f^{\mathsf{T}}};\vartheta^{\mathsf{T}}_f;\vartheta_f$,

ii) $\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}};\mathbb{T} \cap \vartheta_f = \vartheta^{\mathsf{T}}_{f^{\mathsf{T}}} \cap \mathbb{T};\vartheta_f$,

iii) $\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}} \subseteq \vartheta_f$ or $\mathtt{syq}(\varepsilon_X, f;\varepsilon_Y) \subseteq \mathtt{syq}(f^{\mathsf{T}};\varepsilon_X, \varepsilon_Y)$ when f is surjective.

Proof: i) and ii) are proved together, starting with

$\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}};\mathbb{T} \cap \vartheta_f \subseteq (\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}} \cap \vartheta_f;\mathbb{T});(\mathbb{T} \cap \vartheta_{f^{\mathsf{T}}};\vartheta_f)$ Dedekind rule
$= \vartheta^{\mathsf{T}}_{f^{\mathsf{T}}};\vartheta_{f^{\mathsf{T}}};\vartheta_f$ since existential images are total
$\subseteq \vartheta^{\mathsf{T}}_{f^{\mathsf{T}}};\mathbb{T} \cap \vartheta_f$ since existential images are univalent

resulting in equality in between. Similarly

$\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}} \cap \mathbb{T};\vartheta_f \subseteq (\mathbb{T} \cap \vartheta^{\mathsf{T}}_{f^{\mathsf{T}}};\vartheta^{\mathsf{T}}_f);(\vartheta_f \cap \mathbb{T};\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}})$
$= \vartheta^{\mathsf{T}}_{f^{\mathsf{T}}};\vartheta^{\mathsf{T}}_f;\vartheta_f$
$\subseteq \vartheta^{\mathsf{T}}_{f^{\mathsf{T}}} \cap \mathbb{T};\vartheta_f$

Thus, (i,ii) mean the same. For the remaining proof we start from

$\vartheta^{\mathsf{T}}_f;(\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}} \cap \mathbb{T};\vartheta_f) = \mathbb{T};\vartheta_f \cap \vartheta^{\mathsf{T}}_f;\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}}$ masking
$\subseteq (\mathbb{T} \cap \vartheta^{\mathsf{T}}_f;\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}};\vartheta^{\mathsf{T}}_f);(\vartheta_f \cap \mathbb{T};\vartheta^{\mathsf{T}}_f;\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}})$ Dedekind rule
$\subseteq \vartheta^{\mathsf{T}}_f;\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}};\vartheta^{\mathsf{T}}_f;\vartheta_f$
$= \vartheta^{\mathsf{T}}_{f;f^{\mathsf{T}};f};\vartheta_f$ existential images are multiplicative
$= \vartheta^{\mathsf{T}}_f;\vartheta_f \subseteq \mathbb{I}$ since $f;f^{\mathsf{T}};f = f$ for a mapping

Now shunting gives the needed result $\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}} \cap \mathbb{T};\vartheta_f \subseteq \vartheta_f$.

The following is proved mutatis mutandis:

$(\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}};\mathbb{T} \cap \vartheta_f);\vartheta_{f^{\mathsf{T}}} = \vartheta^{\mathsf{T}}_{f^{\mathsf{T}}};\mathbb{T} \cap \vartheta_f;\vartheta_{f^{\mathsf{T}}} \subseteq (\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}} \cap \vartheta_f;\vartheta_{f^{\mathsf{T}}};\mathbb{T});(\mathbb{T} \cap \vartheta_{f^{\mathsf{T}}};\vartheta_f;\vartheta_{f^{\mathsf{T}}})$
$= (\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}} \cap \vartheta_f;\vartheta_{f^{\mathsf{T}}};\mathbb{T});\vartheta_{f^{\mathsf{T}}};\vartheta_f;\vartheta_{f^{\mathsf{T}}} \subseteq \vartheta^{\mathsf{T}}_{f^{\mathsf{T}}};\vartheta_{f^{\mathsf{T}}};\vartheta_f;\vartheta_{f^{\mathsf{T}}}$
$= \vartheta^{\mathsf{T}}_{f^{\mathsf{T}}};\vartheta_{f^{\mathsf{T}};f;f^{\mathsf{T}}} = \vartheta^{\mathsf{T}}_{f^{\mathsf{T}}};\vartheta_{f^{\mathsf{T}}} \subseteq \mathbb{I}$

iii) results simply from an application of Prop. 3.6. □

These results imply not least that $\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}}$ is univalent, or a partial function, when f is surjective. With (ii), we have then also $\vartheta_f \cap \vartheta^{\mathsf{T}}_{f^{\mathsf{T}}};\mathbb{T} = \vartheta^{\mathsf{T}}_{f^{\mathsf{T}}}$.

$$
f = \begin{array}{r c}
 & \begin{array}{cccc} a & b & c & d \end{array} \\
\begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\end{array}
$$

$$
\vartheta_f = \begin{array}{r c}
 & \begin{array}{cccccccccccccccc} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \end{array} \\
\begin{array}{r} \{\} \\ \{1\} \\ \{2\} \\ \{1,2\} \\ \{3\} \\ \{1,3\} \\ \{2,3\} \\ \{1,2,3\} \\ \{4\} \\ \{1,4\} \\ \{2,4\} \\ \{1,2,4\} \\ \{3,4\} \\ \{1,3,4\} \\ \{2,3,4\} \\ \{1,2,3,4\} \\ \{5\} \\ \{1,5\} \\ \{2,5\} \\ \{1,2,5\} \\ \{3,5\} \\ \{1,3,5\} \\ \{2,3,5\} \\ \{1,2,3,5\} \\ \{4,5\} \\ \{1,4,5\} \\ \{2,4,5\} \\ \{1,2,4,5\} \\ \{3,4,5\} \\ \{1,3,4,5\} \\ \{2,3,4,5\} \\ \{1,2,3,4,5\} \end{array} &
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}
$$

$$
\vartheta^\mathsf{T}_{f^\mathsf{T}} = \begin{array}{c}
\begin{array}{cccccccccccccccc} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}
$$

Fig. 5.2 Existential and inverse image for a surjective mapping

$$
f = \begin{array}{r c}
 & \begin{array}{cccc} a & b & c & d \end{array} \\
\begin{array}{r} 1 \\ 2 \\ 3 \end{array} &
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

$$
\vartheta_f = \begin{array}{r c}
 & \begin{array}{cccccccccccccccc} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \end{array} \\
\begin{array}{r} \{\} \\ \{1\} \\ \{2\} \\ \{1,2\} \\ \{3\} \\ \{1,3\} \\ \{2,3\} \\ \{1,2,3\} \end{array} &
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

$$
\vartheta^\mathsf{T}_{f^\mathsf{T}} = \begin{array}{c}
\begin{array}{cccccccccccccccc} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}
$$

Fig. 5.3 Existential and inverse image for a non-surjective mapping

5.5 Proposition. Consider a relation $R : X \longrightarrow Y$ as well as the corresponding power relator $\zeta_R := \big(\varepsilon\backslash(R;\varepsilon')\big) \cap \big((\varepsilon^\mathsf{T};R)/\varepsilon'^\mathsf{T}\big) = \overline{\varepsilon^\mathsf{T};\overline{R;\varepsilon'}} \cap \overline{\overline{\varepsilon^\mathsf{T};R};\varepsilon'} : \mathbf{2}^X \longrightarrow \mathbf{2}^Y$. Then

i) R univalent $\implies$ ζ_R univalent

ii) R surjective $\implies$ ζ_R surjective

iii) R total $\implies$ ζ_R total

iv) R injective $\implies$ ζ_R injective

v) f mapping $\implies$ $\zeta_f = \vartheta_{f}$

Proof: The proofs of (i,. . . ,iv) follow all the same scheme using Prop. 19.11 of [Sch11].

$$\zeta_R^\mathsf{T};\zeta_R = \zeta_{R^\mathsf{T}};\zeta_R = \zeta_{R^\mathsf{T};R} = \overline{\varepsilon'^\mathsf{T};\overline{R^\mathsf{T};R;\varepsilon'}} \cap \overline{\overline{\varepsilon'^\mathsf{T};R^\mathsf{T};R};\varepsilon'}$$
$$\subseteq \overline{\varepsilon'^\mathsf{T};\overline{\mathbb{I};\varepsilon'}} \cap \overline{\overline{\varepsilon'^\mathsf{T};\mathbb{I}};\varepsilon'} = \overline{\varepsilon'^\mathsf{T};\overline{\varepsilon'}} \cap \overline{\overline{\varepsilon'^\mathsf{T}};\varepsilon'} = \Omega' \cap \Omega'^\mathsf{T} = \mathbb{I}$$

v) $\zeta_f = \overline{\varepsilon^\mathsf{T};\overline{f;\varepsilon'}} \cap \overline{\overline{\varepsilon^\mathsf{T};f};\varepsilon'} = \overline{\varepsilon^\mathsf{T};f;\overline{\varepsilon'}} \cap \overline{\overline{\varepsilon^\mathsf{T};f};\varepsilon'} = \mathtt{syq}(f^\mathsf{T};\varepsilon,\varepsilon') = \vartheta_{f}$ □

The construct ζ looks quite similar to a symmetric quotient, but it is not!

6 Relations in varying representations

When dealing with relations, we have — in principle — three incarnations of the same idea. A relation between sets X, Y may, namely, be represented

- as $R : X \longrightarrow Y$ corresponding to a possibly non-square Boolean matrix,
- as $\mathfrak{r} : X \times Y \longrightarrow \mathbb{1}$ corresponding to a Boolean vector characterizing a subset of pairs,
- as $r : \mathbf{2}^{X\times Y} \longrightarrow \mathbb{1}$ corresponding to a point in the powerset of the pair set.

Their interrelationship using projections $\pi : X \times Y \longrightarrow X$, resp. $\rho : X \times Y \longrightarrow Y$, and the membership relation $\varepsilon_\times : X \times Y \longrightarrow \mathbf{2}^{X\times Y}$ starting in the product is as follows:

$$\mathfrak{r} = \varepsilon_\times;r \qquad r = \mathtt{syq}(\varepsilon_\times, \mathfrak{r}) \qquad R = \mathtt{rel}(\mathfrak{r}) = \pi^\mathsf{T};(\mathfrak{r};\mathbb{T}_{\mathbb{1},Y} \cap \rho) \qquad \mathfrak{r} = \mathtt{vec}(R) = (\pi;R \cap \rho);\mathbb{T}_{Y,\mathbb{1}}$$

The transition from R to $\mathfrak{r}$ is a **vectorization**, known also at other occasions in algebra. While it may be considered an easy construction, one should think of a 5000×1000-relation and its vectorization that may be much harder to handle in practice.

6.1 Proposition. $R = \mathtt{rel}(\mathtt{vec}(R))$ and $\mathfrak{r} = \mathtt{vec}(\mathtt{rel}(\mathfrak{r}))$

Proof: $R = \pi^\mathsf{T};\rho;\mathbb{I}_Y \cap R$

$\subseteq (\pi^\mathsf{T};\rho \cap R;\mathbb{I}_Y);(\mathbb{I}_Y \cap (\pi^\mathsf{T};\rho)^\mathsf{T};R)$ Dedekind

$= \pi^\mathsf{T};(\rho \cap \pi;R);(\mathbb{I}_Y \cap \mathbb{T}_{Y,X};R)$

$= \pi^\mathsf{T};[(\rho \cap \pi;R);\mathbb{I}_Y \cap (\rho \cap \pi;R);\mathbb{T}_{Y,X};R)]$ since $(\rho \cap \pi;R)$ is univalent

$= \pi^\mathsf{T};[\rho \cap \pi;R \cap (\rho \cap \pi;R);\mathbb{T}_{Y,X};R)]$

$\subseteq \pi^\mathsf{T};[\rho \cap (\rho \cap \pi;R);\mathbb{T}_{Y,Y})] = \pi^\mathsf{T};[(\pi;R \cap \rho);\mathbb{T}_{Y,\mathbb{1}};\mathbb{T}_{\mathbb{1},Y}) \cap \rho] = \mathtt{rel}(\mathtt{vec}(R))$

$\subseteq \pi^\mathsf{T};((\pi;R \cap \rho) \cap \rho;\mathbb{T}_{Y,Y});(\mathbb{T}_{Y,Y} \cap (\pi;R \cap \rho)^\mathsf{T};\rho)$ Dedekind rule

$\subseteq \pi^\mathsf{T};\pi;R;\rho^\mathsf{T};\rho = R$

$$\begin{aligned}
&\mathfrak{r} = (\mathbb{T}_{X\times Y,\mathbb{1}} \cap \mathfrak{r}){:}\mathbb{T}_{\mathbb{1},\mathbb{1}} = (\rho{:}\mathbb{T}_{Y,\mathbb{1}} \cap \mathfrak{r}){:}\mathbb{T}_{\mathbb{1},\mathbb{1}}\\
&\subseteq (\rho \cap \mathfrak{r}{:}\mathbb{T}_{\mathbb{1},Y}){:}(\mathbb{T}_{Y,\mathbb{1}} \cap \rho^{\mathsf{T}}{:}\mathfrak{r}){:}\mathbb{T}_{\mathbb{1},\mathbb{1}}\\
&\subseteq (\rho \cap \mathfrak{r}{:}\mathbb{T}_{\mathbb{1},Y}){:}\mathbb{T}_{Y,\mathbb{1}}\\
&= ((\mathfrak{r}{:}\mathbb{T}_{\mathbb{1},Y} \cap \rho) \cap \rho){:}\mathbb{T}_{Y,\mathbb{1}}\\
&\subseteq \left[\pi{:}\pi^{\mathsf{T}}{:}(\mathfrak{r}{:}\mathbb{T}_{\mathbb{1},Y} \cap \rho) \cap \rho\right]{:}\mathbb{T}_{Y,\mathbb{1}} = \mathtt{vec}(\mathtt{rel}(\mathfrak{r}))\\
&\subseteq \left[\pi{:}\pi^{\mathsf{T}} \cap \rho{:}(\mathfrak{r}{:}\mathbb{T}_{\mathbb{1},Y} \cap \rho)^{\mathsf{T}}\right]{:}\left[(\mathfrak{r}{:}\mathbb{T}_{\mathbb{1},Y} \cap \rho) \cap \pi{:}\pi^{\mathsf{T}}{:}\rho\right]{:}\mathbb{T}_{Y,\mathbb{1}}\\
&\subseteq \left[\pi{:}\pi^{\mathsf{T}} \cap \rho{:}\rho^{\mathsf{T}}\right]{:}\mathfrak{r}{:}\mathbb{T}_{\mathbb{1},Y}{:}\mathbb{T}_{Y,\mathbb{1}}\\
&= \mathfrak{r}{:}\mathbb{T}_{\mathbb{1},Y}{:}\mathbb{T}_{Y,\mathbb{1}} = \mathfrak{r}{:}\mathbb{T}_{\mathbb{1},\mathbb{1}} = \mathfrak{r} \qquad \square
\end{aligned}$$

It should be made clear that the relations with standard abbreviation $\varepsilon_\times, \pi, \rho$ do not fall from heaven. Rather, they are defined generically as characterizations *up to isomorphism* using the techniques of domain construction developed in [Sch11]. They allow to formulate via a language called TituRel

$\pi \approx \mathtt{Pi}\ \ X\ Y \qquad \rho \approx \mathtt{Rho}\ X\ Y \qquad$ given that $X = \mathtt{src}(R)$ and $Y = \mathtt{tgt}(R)$
$\varepsilon_\times \approx \mathtt{ElemIn\ (DirPro}\ \ X\ Y\mathtt{)}$

Following the idea of the threefold ways of denoting, the identity $\mathbb{I} : X \longrightarrow X$ gives rise to the vector $\mathtt{vec}(\mathbb{I}) = (\pi \cap \rho){:}\mathbb{T} : X \times X \longrightarrow \mathbb{1}$ and finally to the element $\mathcal{I} = \mathtt{syq}(\varepsilon_\times, (\pi \cap \rho){:}\mathbb{T}) : \mathbf{2}^{X\times X} \longrightarrow \mathbb{1}$ in the powerset of all pairs.

6.2 Proposition. Consider a set X together with the membership $\varepsilon_\times : X \times X \longrightarrow \mathbf{2}^{X\times X}$ on the direct product of the set with itself and define the point

$\mathcal{I} := \mathtt{syq}(\varepsilon_\times, (\pi \cap \rho){:}\mathbb{T}) = \mathtt{syq}(\varepsilon_\times, \mathtt{vec}(\mathbb{I})).$

Then $\mathtt{rel}(\varepsilon_\times{:}\mathcal{I}) = \mathbb{I}$.

Proof: $\mathtt{rel}(\varepsilon_\times{:}\mathcal{I}) = \mathtt{rel}(\varepsilon_\times{:}\mathtt{syq}(\varepsilon_\times, (\pi \cap \rho){:}\mathbb{T})) = \mathtt{rel}((\pi \cap \rho){:}\mathbb{T})$
$= \pi^{\mathsf{T}}{:}\left[(\pi \cap \rho){:}\mathbb{T} \cap \rho\right]$ expanded
$= \pi^{\mathsf{T}}{:}(\pi \cap \rho)$ see below
$= \mathbb{I} \cap \pi^{\mathsf{T}}{:}\rho = \mathbb{I} \cap \mathbb{T} = \mathbb{I}$

Now the postponed transition is justified with a sequence of containments implying equality:

$(\pi \cap \rho){:}\mathbb{T} \cap \rho \subseteq [(\pi \cap \rho) \cap \rho{:}\mathbb{T}]{:}[\mathbb{T} \cap (\pi \cap \rho)^{\mathsf{T}}{:}\rho] = (\pi \cap \rho){:}(\pi^{\mathsf{T}} \cap \rho^{\mathsf{T}}){:}\rho \qquad \rho$ is total
$= (\pi \cap \rho){:}(\pi^{\mathsf{T}}{:}\rho \cap \mathbb{I}) = (\pi \cap \rho){:}(\mathbb{T} \cap \mathbb{I}) = \pi \cap \rho \subseteq (\pi \cap \rho){:}\mathbb{T} \cap \rho \qquad \square$

Much in the same way as later for $\mathfrak{M}$, $\mathfrak{J}$, we show here that it is possible to express the least and the greatest relations as points

$\mathcal{BOT} : \mathbf{2}^{X\times Y} \longrightarrow \mathbb{1}, \qquad \mathcal{TOP} : \mathbf{2}^{X\times Y} \longrightarrow \mathbb{1}.$

6.3 Proposition. Consider sets X, Y together with the membership $\varepsilon_\times : X \times Y \longrightarrow \mathbf{2}^{X\times Y}$ on the direct product of the sets and define the point

$\mathcal{BOT} := \mathtt{syq}(\varepsilon_\times, \mathtt{vec}(\mathbb{\perp})) = \mathtt{syq}(\varepsilon_\times, \mathbb{\perp})$
$\mathcal{TOP} := \mathtt{syq}(\varepsilon_\times, \mathtt{vec}(\mathbb{T})) = \mathtt{syq}(\varepsilon_\times, \mathbb{T}).$

Then $\mathtt{rel}(\varepsilon_\times{:}\mathcal{BOT}) = \mathbb{\perp}$ and $\mathtt{rel}(\varepsilon_\times{:}\mathcal{TOP}) = \mathbb{T}$.

Proof: $\mathtt{rel}(\varepsilon_\times; \mathcal{BOT}) = \mathtt{rel}(\varepsilon_\times; \mathtt{syq}(\varepsilon_\times, \mathtt{vec}(\perp\!\!\!\perp))) = \mathtt{rel}(\mathtt{vec}(\perp\!\!\!\perp)) = \perp\!\!\!\perp$ $\square$

The processes of transposition and negation

$$\mathcal{T} : \mathbf{2}^{X\times Y} \longrightarrow \mathbf{2}^{Y\times X} \qquad N : \mathbf{2}^{X\times Y} \longrightarrow \mathbf{2}^{X\times Y},$$

may also be conceived as bijective mappings, as well as the process of composition

$$\mathcal{C} : \mathbf{2}^{X\times Y} \times \mathbf{2}^{Y\times Z} \longrightarrow \mathbf{2}^{X\times Z},$$

as a binary mapping, i.e., all three in a pointfree fashion. While we omit discussing $\mathcal{C}$, we refer for N to Fig. 9.3. Here, we restrict to studying formally the interchange of components of a pair, which obviously determines a bijective mapping

$$\mathcal{T} : \mathbf{2}^{X\times Y} \longrightarrow \mathbf{2}^{Y\times X},$$

satisfying certain rules.

$$\begin{tikzcd}
2^{X\times Y} \arrow[r, "\mathcal{T}"] & 2^{Y\times X} \\
X\times Y \arrow[u, "\varepsilon"] \arrow[r, "P"] \arrow[d, "\pi"'] \arrow[dr, "\rho"] & Y\times X \arrow[u, "\varepsilon'"'] \arrow[d, "\pi'"] \arrow[dl, "\rho'"'] \\
X \arrow[r, "R"] & Y \arrow[l, bend left, "R^{\mathsf{T}}"]
\end{tikzcd}$$

Fig. 6.1 Illustrating transposition as an operation on relations conceived as points

6.4 Proposition. Consider two sets X, Y together with the memberships $\varepsilon : X \times Y \longrightarrow \mathbf{2}^{X\times Y}$ and $\varepsilon' : Y \times X \longrightarrow \mathbf{2}^{Y\times X}$ of both their direct products and define

$$P := \pi; \rho'^{\mathsf{T}} \cap \rho; \pi'^{\mathsf{T}} \qquad \mathcal{T} := \mathtt{syq}(P^{\mathsf{T}}; \varepsilon, \varepsilon').$$

Then

i) P is a bijective mapping to be interpreted as sending (x, y) to (y, x).

ii) $\mathcal{T}$ is a bijective mapping resembling transposition.

iii) $P; \rho' = \pi \qquad P; \pi' = \rho \qquad P^{\mathsf{T}}; \pi = \rho' \qquad P^{\mathsf{T}}; \rho = \pi'$

iv) $\mathcal{T}; \varepsilon'^{\mathsf{T}} = \varepsilon^{\mathsf{T}}; P$, i.e., P and $\mathcal{T}$ bisimulate one another via the membership relations.

v) $\mathtt{rel}(P^{\mathsf{T}}; v) = \left[\mathtt{rel}(v)\right]^{\mathsf{T}}$

vi) $\mathtt{vec}(R^{\mathsf{T}}) = P; \mathtt{vec}(R)$

vii) $\mathtt{rel}(\overline{v}) = \overline{\mathtt{rel}(v)}$

viii) $\mathtt{vec}(\overline{R}) = \overline{\mathtt{vec}(R)}$

Proof: i) $P^{\mathsf T};P = (\rho';\pi^{\mathsf T} \cap \pi';\rho^{\mathsf T});(\pi;\rho'^{\mathsf T} \cap \rho;\pi'^{\mathsf T}) \subseteq \rho'\,;\pi^{\mathsf T};\pi;\rho'^{\mathsf T} \cap \pi'\,;\rho^{\mathsf T};\rho;\pi'^{\mathsf T} = \rho'\,;\rho'^{\mathsf T} \cap \pi'\,;\pi'^{\mathsf T} = \mathbb{I}$

This shows univalency; analogously for injectivity. Therefore P multiplies distributively over conjunction and we may proceed with

$$
\begin{aligned}
&P;P^{\mathsf T} = (\pi;\rho'^{\mathsf T} \cap \rho;\pi'^{\mathsf T});(\rho';\pi^{\mathsf T} \cap \pi';\rho^{\mathsf T}) = (\pi;\rho'^{\mathsf T} \cap \rho;\pi'^{\mathsf T});\rho';\pi^{\mathsf T} \cap (\pi;\rho'^{\mathsf T} \cap \rho;\pi'^{\mathsf T});\pi';\rho^{\mathsf T}\\
&= (\pi \cap \rho;\pi'^{\mathsf T};\rho');\pi^{\mathsf T} \cap (\pi;\rho'^{\mathsf T};\pi' \cap \rho);\rho^{\mathsf T}\\
&= (\pi \cap \mathbb{T});\pi^{\mathsf T} \cap (\mathbb{T} \cap \rho);\rho^{\mathsf T} = \pi;\pi^{\mathsf T} \cap \rho;\rho^{\mathsf T} = \mathbb{I},
\end{aligned}
$$

giving totality, and in analogy also surjectivity.

ii) $\mathcal{T}$ is univalent, since $\mathcal{T}^{\mathsf T};\mathcal{T} = \mathtt{syq}(\varepsilon', P^{\mathsf T};\varepsilon);\mathtt{syq}(P^{\mathsf T};\varepsilon, \varepsilon') = \mathtt{syq}(\varepsilon', \varepsilon') = \mathbb{I}$. It is total because $\mathcal{T}^{\mathsf T} = \mathtt{syq}(\varepsilon', P^{\mathsf T};\varepsilon)$ is surjective by definition of the membership ε'.

iii) is trivial.

iv) $\mathcal{T};\varepsilon'^{\mathsf T} = [\varepsilon';\mathcal{T}^{\mathsf T}]^{\mathsf T} = [\varepsilon';\mathtt{syq}(\varepsilon', P^{\mathsf T};\varepsilon)]^{\mathsf T} = [P^{\mathsf T};\varepsilon]^{\mathsf T} = \varepsilon^{\mathsf T};P$

v) $\mathtt{rel}(P^{\mathsf T};v) = \pi'^{\mathsf T};(P^{\mathsf T};v;\mathbb{T} \cap \rho') = \pi'^{\mathsf T};(P^{\mathsf T};v;\mathbb{T} \cap P^{\mathsf T};\pi) = \pi'^{\mathsf T};P^{\mathsf T};(v;\mathbb{T} \cap \pi) = \rho^{\mathsf T};(v;\mathbb{T} \cap \pi)$
$= \big[(v;\mathbb{T} \cap \pi)^{\mathsf T};\rho\big]^{\mathsf T} = \big[(\mathbb{T};v^{\mathsf T} \cap \pi^{\mathsf T});\rho\big]^{\mathsf T} = \big[\pi^{\mathsf T}(v;\mathbb{T} \cap \rho)\big]^{\mathsf T} = \big[\mathtt{rel}(v)\big]^{\mathsf T}$

vi) $\mathtt{vec}(R^{\mathsf T}) = (\pi';R^{\mathsf T} \cap \rho');\mathbb{T} = (P^{\mathsf T};\rho;R^{\mathsf T} \cap P^{\mathsf T};\pi);\mathbb{T} = P^{\mathsf T};(\rho;R^{\mathsf T} \cap \pi);\mathbb{T}$
$\subseteq P^{\mathsf T};(\rho \cap \pi;R);(R^{\mathsf T} \cap \rho^{\mathsf T};\pi);\mathbb{T} = P^{\mathsf T};(\rho \cap \pi;R);(R^{\mathsf T} \cap \mathbb{T});\mathbb{T} \subseteq P^{\mathsf T};(\rho \cap \pi;R);\mathbb{T} = P^{\mathsf T};(\pi;R \cap \rho);\mathbb{T}$
$\subseteq P^{\mathsf T};(\pi \cap \rho;R^{\mathsf T});(R \cap \pi^{\mathsf T}\rho);\mathbb{T} \subseteq P^{\mathsf T};(\pi \cap \rho;R^{\mathsf T});\mathbb{T}$ implying equality everywhere in between
$= P^{\mathsf T};\mathtt{vec}(R)$

vii) $\mathbb{T} = \pi^{\mathsf T};\rho = \pi^{\mathsf T};(\rho \cap \mathfrak{r};\mathbb{T}) \cup \pi^{\mathsf T};(\rho \cap \overline{\mathfrak{r};\mathbb{T}}) \quad\Longrightarrow\quad \overline{\mathtt{rel}(\mathfrak{r})} = \overline{\pi^{\mathsf T};(\rho \cap \mathfrak{r};\mathbb{T})} \subseteq \pi^{\mathsf T};(\rho \cap \overline{\mathfrak{r};\mathbb{T}}) = \mathtt{rel}(\bar{\mathfrak{r}})$
$\pi;\pi^{\mathsf T};(\rho \cap \mathfrak{r};\mathbb{T}) \cap \rho \subseteq (\pi;\pi^{\mathsf T} \cap \rho;(\rho \cap \mathfrak{r};\mathbb{T})^{\mathsf T});(\rho \cap \mathfrak{r};\mathbb{T} \cap \pi;\pi^{\mathsf T};\rho) \subseteq \mathbb{I};\mathfrak{r};\mathbb{T} = \mathfrak{r};\mathbb{T}$ Dedekind rule
$\Longrightarrow \quad \pi;\pi^{\mathsf T};(\rho \cap \mathfrak{r};\mathbb{T}) \subseteq \overline{\rho} \cup \mathfrak{r};\mathbb{T} \quad\Longleftrightarrow\quad \mathtt{rel}(\bar{\mathfrak{r}}) = \pi^{\mathsf T};(\rho \cap \overline{\mathfrak{r};\mathbb{T}}) \subseteq \overline{\pi^{\mathsf T};(\rho \cap \mathfrak{r};\mathbb{T})} = \overline{\mathtt{rel}(\mathfrak{r})}$

viii) $\mathtt{vec}(\overline{R}) = \mathtt{vec}(\overline{\mathtt{rel}(\mathtt{vec}(R))})$ Prop. 6.1
$= \mathtt{vec}(\mathtt{rel}(\overline{\mathtt{vec}(R)}))$ according to (vii)
$= \overline{\mathtt{vec}(R)}$ Prop. 6.1 □

7 Some categorical considerations

We here give relation-algebraic proofs of certain results we will use afterwards. Everything is fully based on the generic constructions of a direct sum, or product, etc. If any two heterogeneous relations π, ρ with common source are given, they are said to form a **direct product** if

$$\pi^{\mathsf T};\pi = \mathbb{I}, \quad \rho^{\mathsf T};\rho = \mathbb{I}, \quad \pi;\pi^{\mathsf T} \cap \rho;\rho^{\mathsf T} = \mathbb{I}, \quad \pi^{\mathsf T};\rho = \mathbb{T}.$$

Thus, the relations π, ρ are mappings, usually called **projections**. In a similar way, any two heterogeneous relations ι, κ with common target are said to form the left, respectively right, **injection** of a **direct sum** if

$$\iota;\iota^{\mathsf T} = \mathbb{I}, \quad \kappa;\kappa^{\mathsf T} = \mathbb{I}, \quad \iota^{\mathsf T};\iota \cup \kappa^{\mathsf T};\kappa = \mathbb{I}, \quad \iota;\kappa^{\mathsf T} = \mathbb{⊥}.$$

7.1 Definition. Given any two direct products by projections

$$\pi : X \times Y \longrightarrow X, \quad \rho : X \times Y \longrightarrow Y, \qquad \pi' : U \times V \longrightarrow U, \quad \rho' : U \times V \longrightarrow V,$$

we define as binary operations on relations

i) $(A \otimes B) := \pi{;}A{;}\pi'^{\mathsf{T}} \cap \rho{;}B{;}\rho'^{\mathsf{T}} : \quad X \times Y \longrightarrow U \times V$, the **Kronecker product**,

ii) $(C \olessthan D) := C{;}\pi^{\mathsf{T}} \cap D{;}\rho^{\mathsf{T}} : \quad Z \longrightarrow X \times Y$, the **fork-operator**,

iii) $(E \ogreaterthan F) := \pi{;}E \cap \rho{;}F : \quad X \times Y \longrightarrow W$, the **join-operator**. □

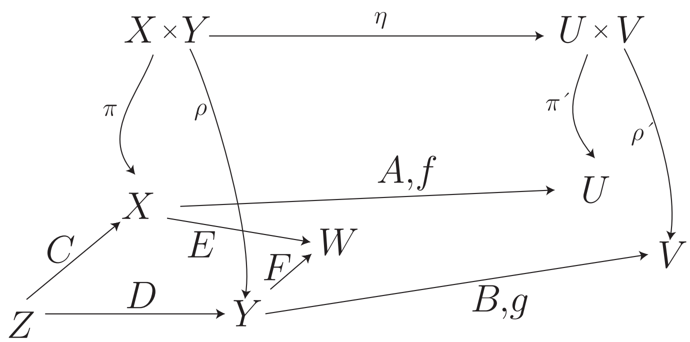

Fig. 7.1 Kronecker, fork-, and join-operators applied to relations and mappings

Obvious identities are $(A \otimes B)^{\mathsf{T}} = (A^{\mathsf{T}} \otimes B^{\mathsf{T}})$ and $(C \olessthan D)^{\mathsf{T}} = (C^{\mathsf{T}} \ogreaterthan D^{\mathsf{T}})$. The next results are presented in some detail because they are very close to the 'unsharpness' situation where model problems arise: There exist relational formulae that hold in the classical interpretation, but cannot be derived in the axiomatization followed here; see [Sch11, Sect. 7.2].

7.2 Proposition. Let be given any two direct products by projections

$\pi : X \times Y \longrightarrow X, \quad \rho : X \times Y \longrightarrow Y, \qquad \pi' : U \times V \longrightarrow U, \quad \rho' : U \times V \longrightarrow V$

together with relations $A : X \longrightarrow U$ and $B : Y \longrightarrow V$. Then

i) $(A \otimes B){;}\pi' = \pi{;}A \cap \rho{;}B{;}\mathbb{T} \qquad (A \otimes B){;}\rho' = \pi{;}A{;}\mathbb{T} \cap \rho{;}B$

ii) $(A \otimes B){;}\pi' = \pi{;}A$ in case B is total

iii) $(A \otimes B){;}\rho' = \rho{;}B$ in case A is total

iv) $(A \otimes \mathbb{I}){;}\pi' = (A \ogreaterthan \mathbb{T}) = \pi{;}A \qquad (\mathbb{I} \otimes B){;}\rho' = (\mathbb{T} \ogreaterthan B) = \rho{;}B.$

$(A \olessthan \mathbb{I}){;}\pi' = A \qquad (\mathbb{I} \olessthan B){;}\rho' = B.$

v) If A, B are both univalent, then so is $(A \otimes B)$.

vi) If A, B are both mappings, then so is $(A \otimes B)$.

Proof: i) $(A \otimes B){;}\pi' = (\pi{;}A{;}\pi'^{\mathsf{T}} \cap \rho{;}B{;}\rho'^{\mathsf{T}}){;}\pi'$ by definition
$= \pi{;}A \cap \rho{;}B{;}\rho'^{\mathsf{T}}{;}\pi'$ since π' is univalent, [Sch11], Prop. 5.4
$= \pi{;}A \cap \rho{;}B{;}\mathbb{T}$ property of the direct product π', ρ'
The second formula is derived analogously.

ii) and (iii) are trivial consequences.

iv) $(\mathbb{I} \otimes B);\rho' = (\pi;\pi'^{\mathsf{T}} \cap \rho;B;\rho'^{\mathsf{T}});\rho' = \pi;\pi'^{\mathsf{T}};\rho' \cap \rho;B = \pi;\mathbb{T} \cap \rho;B = (\mathbb{T} \ogreaterthan B) = \mathbb{T} \cap \rho;B$

v) $(A \otimes B)^{\mathsf{T}};(A \otimes B) = (\pi';A^{\mathsf{T}};\pi^{\mathsf{T}} \cap \rho';B^{\mathsf{T}};\rho^{\mathsf{T}});(\pi;A;\pi'^{\mathsf{T}} \cap \rho;B;\rho'^{\mathsf{T}})$ by definition
$\subseteq \pi';A^{\mathsf{T}};\pi^{\mathsf{T}};\pi;A;\pi'^{\mathsf{T}} \cap \rho';B^{\mathsf{T}};\rho^{\mathsf{T}};\rho;B;\rho'^{\mathsf{T}}$ monotony
$\subseteq \pi';A^{\mathsf{T}};A;\pi'^{\mathsf{T}} \cap \rho';B^{\mathsf{T}};B;\rho'^{\mathsf{T}}$ since projections π, ρ are univalent
$\subseteq \pi';\pi'^{\mathsf{T}} \cap \rho';\rho'^{\mathsf{T}}$ since A, B are assumed to be univalent
$= \mathbb{I}$ by definition of a direct product

vi) Univalency follows from (iv). $(A \otimes B);\mathbb{T} \supseteq (A \otimes B);\pi';\mathbb{T} = \pi;A;\mathbb{T} = \pi;\mathbb{T} = \mathbb{T}$ □

The results above are more or less known. It was important to execute rigorous axiomatic proofs, i.e., not just based on Boolean matrices. Of course, analogous formulae hold in the converse situation.

7.3 Proposition. Let be given the setting above.

i) $(R \otimes S);(P \otimes Q) \subseteq (R;P \otimes S;Q)$

$(R \otimes S);(P \ogreaterthan Q) \subseteq (R;P \ogreaterthan S;Q)$ $\qquad (R \olessthan S);(P \otimes Q) \subseteq (R;P \olessthan S;Q)$

$(R \olessthan S);(P \ogreaterthan Q) \subseteq R;P \cap S;Q$

ii) $(f \otimes g);(A \otimes B) = (f;A \otimes g;B)$ provided f, g are both univalent

iii) $(f \otimes g);(A \ogreaterthan B) = (f;A \ogreaterthan g;B)$ provided f, g are both univalent

$(R \otimes S);(A \ogreaterthan B) = (R;A \ogreaterthan S;B)$ provided A, B are both injective

$(R \olessthan S);(A \ogreaterthan B) = R;A \cap S;B$ provided A, B are both injective, or R, S both univalent

iv) $(R \otimes S);\mathbb{T} = (R;\mathbb{T} \ogreaterthan S;\mathbb{T})$

v) $(A \olessthan B) \cap C;\mathbb{T} = (A \cap C;\mathbb{T} \olessthan B \cap C;\mathbb{T})$

vi) $(A \otimes B) \cap (C;\mathbb{T} \ogreaterthan D;\mathbb{T}) = (A \cap C;\mathbb{T} \otimes B \cap D;\mathbb{T})$

vii) $(A \cap C \olessthan B \cap D) = (A \olessthan B) \cap (C \olessthan D)$

viii) $(R\backslash R \otimes S\backslash S) \subseteq (R \olessthan S) \backslash (R \olessthan S)$

ix) $C;(A \olessthan B) = (C;A \olessthan C;B)$ provided C is univalent

Proof: i) See [Sch11, Prop. 7.2.ii], where it is also mentioned that a pointfree proof of equality is impossible notwithstanding the fact that equality holds when the Point Axiom is demanded; i.e., not least for Boolean matrices. Indeed, there exist models where equality is violated.

ii) According to Prop. 7.2.iv, $(f \otimes g)$ is univalent, so that we may reason
$(f \otimes g);(A \otimes B) = (f \otimes g);(\pi_2;A;\pi_3^{\mathsf{T}} \cap \rho_2;B;\rho_3^{\mathsf{T}})$ by definition
$= (f \otimes g);\pi_2;A;\pi_3^{\mathsf{T}} \cap (f \otimes g);\rho_2;B;\rho_3^{\mathsf{T}}$ univalency
$= (\pi_1;f \cap \rho_1;g;\mathbb{T});A;\pi_3^{\mathsf{T}} \cap (\pi_1;f;\mathbb{T} \cap \rho_1;g);B;\rho_3^{\mathsf{T}}$ Prop. 7.2.i

$= \pi_1;f;A;\pi_3^{\mathsf{T}} \cap \rho_1;g;\mathbb{T} \;\cap\; \pi_1;f;\mathbb{T} \cap \rho_1;g;B;\rho_3^{\mathsf{T}}$ masking
$= \pi_1;f;A;\pi_3^{\mathsf{T}} \cap \rho_1;g;B;\rho_3^{\mathsf{T}}$ trivial
$= (f;A \otimes g;B)$ by definition

iii) is shown similar to (ii).

iv) For clarity, we mention the ever changing typing of the universal relations explicitly:
$(R \otimes S);\mathbb{T}_{X'\times Y',Z} = (\pi;R;\pi'^{\mathsf{T}} \cap \rho;S;\rho'^{\mathsf{T}});\mathbb{T}_{X'\times Y',Z}$ by definition
$= (\pi;R;\pi'^{\mathsf{T}} \cap \rho;S;\rho'^{\mathsf{T}});\pi';\mathbb{T}_{X',Z}$ since π' is total
$= (\pi;R \cap \rho;S;\rho'^{\mathsf{T}};\pi');\mathbb{T}_{X',Z}$ since π' is univalent, [Sch11], Prop. 5.4
$= (\pi;R \cap \rho;S;\mathbb{T}_{Y',X'});\mathbb{T}_{X',Z}$ property of the direct product
$= \pi;R;\mathbb{T}_{X',Z} \cap \rho;S;\mathbb{T}_{Y',Z}$ masking
$= (R;\mathbb{T}_{X',Z} \ogreaterthan S;\mathbb{T}_{Y',Z})$ by definition

v) $(A \cap C;\mathbb{T}_{Z,X} \olessthan B \cap C;\mathbb{T}_{Z,Y}) = (A \cap C;\mathbb{T}_{Z,X});\pi^{\mathsf{T}} \cap (B \cap C;\mathbb{T}_{Z,Y});\rho^{\mathsf{T}}$
$= A;\pi^{\mathsf{T}} \cap C;\mathbb{T}_{Z,X};\pi^{\mathsf{T}} \cap B;\rho^{\mathsf{T}} \cap C;\mathbb{T}_{Z,Y};\rho^{\mathsf{T}}$
$= A;\pi^{\mathsf{T}} \cap C;\mathbb{T}_{Z,X\times Y} \cap B;\rho^{\mathsf{T}} \cap C;\mathbb{T}_{Z,X\times Y} = A;\pi^{\mathsf{T}} \cap B;\rho^{\mathsf{T}} \cap C;\mathbb{T}_{Z,X\times Y}$
$= (A \olessthan B) \cap C;\mathbb{T}_{Z,X\times Y}$

vi) Assume $A : X \longrightarrow Y, B : U \longrightarrow V, C : X \longrightarrow Z, D : U \longrightarrow W$:
$(A \otimes B) \cap (C;\mathbb{T}_{Z,Y\times V} \ogreaterthan D;\mathbb{T}_{W,Y\times V}) = \pi;A;\pi'^{\mathsf{T}} \cap \rho;B;\rho'^{\mathsf{T}} \cap \pi;C;\mathbb{T}_{Z,Y\times V} \cap \rho;D;\mathbb{T}_{W,Y\times V}$
$= \pi;A;\pi'^{\mathsf{T}} \cap \pi;C;\mathbb{T}_{Z,Y\times V} \cap \rho;B;\rho'^{\mathsf{T}} \cap \rho;D;\mathbb{T}_{W,Y\times V}$ shuffled
$= \pi;(A;\pi'^{\mathsf{T}} \cap C;\mathbb{T}_{Z,Y\times V}) \cap \rho;(B;\rho'^{\mathsf{T}} \cap D;\mathbb{T}_{W,Y\times V})$
$= \pi;(A;\pi'^{\mathsf{T}} \cap C;\mathbb{T}_{Z,Y};\pi'^{\mathsf{T}}) \cap \rho;(B;\rho'^{\mathsf{T}} \cap D;\mathbb{T}_{W,V};\rho'^{\mathsf{T}})$
$= \pi;(A \cap C;\mathbb{T}_{Z,Y});\pi'^{\mathsf{T}} \cap \rho;(B \cap D;\mathbb{T}_{W,V});\rho'^{\mathsf{T}}$
$= (A \cap C;\mathbb{T}_{Z,Y} \otimes B \cap D;\mathbb{T}_{W,V})$

vii) $(A \cap C \olessthan B \cap D) = (A \cap C);\pi^{\mathsf{T}} \cap (B \cap D);\rho^{\mathsf{T}}$
$= A;\pi^{\mathsf{T}} \cap C;\pi^{\mathsf{T}} \cap B;\rho^{\mathsf{T}} \cap D;\rho^{\mathsf{T}}$
$= A;\pi^{\mathsf{T}} \cap B;\rho^{\mathsf{T}} \cap C;\pi^{\mathsf{T}} \cap D;\rho^{\mathsf{T}} = (A \olessthan B) \cap (C \olessthan D)$

viii) $(R \olessthan S) \setminus (R \olessthan S) = \overline{(R \olessthan S)^{\mathsf{T}};\overline{(R \olessthan S)}} = \overline{(R^{\mathsf{T}} \ogreaterthan S^{\mathsf{T}});\overline{R;\pi^{\mathsf{T}} \cap S;\rho^{\mathsf{T}}}}$
$= \overline{(R^{\mathsf{T}} \ogreaterthan S^{\mathsf{T}});(\overline{R;\pi^{\mathsf{T}}} \cup \overline{S;\rho^{\mathsf{T}}})} = \overline{(R^{\mathsf{T}} \ogreaterthan S^{\mathsf{T}});\overline{R;\pi^{\mathsf{T}}} \cup (R^{\mathsf{T}} \ogreaterthan S^{\mathsf{T}});\overline{S;\rho^{\mathsf{T}}}}$
$= \overline{(\pi;R^{\mathsf{T}} \cap \rho;S^{\mathsf{T}});\overline{R};\pi^{\mathsf{T}} \cup (\pi;R^{\mathsf{T}} \cap \rho;S^{\mathsf{T}});\overline{S};\rho^{\mathsf{T}}}$
$\supseteq \overline{\pi;R^{\mathsf{T}};\overline{R};\pi^{\mathsf{T}} \cup \rho;S^{\mathsf{T}};\overline{S};\rho^{\mathsf{T}}}$
$= \overline{\pi;R^{\mathsf{T}};\overline{R};\pi^{\mathsf{T}}} \cap \overline{\rho;S^{\mathsf{T}};\overline{S};\rho^{\mathsf{T}}} = \pi;\overline{R^{\mathsf{T}};\overline{R}};\pi^{\mathsf{T}} \cap \rho;\overline{S^{\mathsf{T}};\overline{S}};\rho^{\mathsf{T}} = (R\backslash R \otimes S\backslash S)$

ix) trivial $\square$

As mentioned, one must not demand *arbitrary* products to exist, because one will then run into model problems. To employ the Point Axiom is a requirement stronger than necessary. When here just two additional products are requested, this means some sort of an “improved observability” for the pairs in the product $A \times B$ via vectorization.

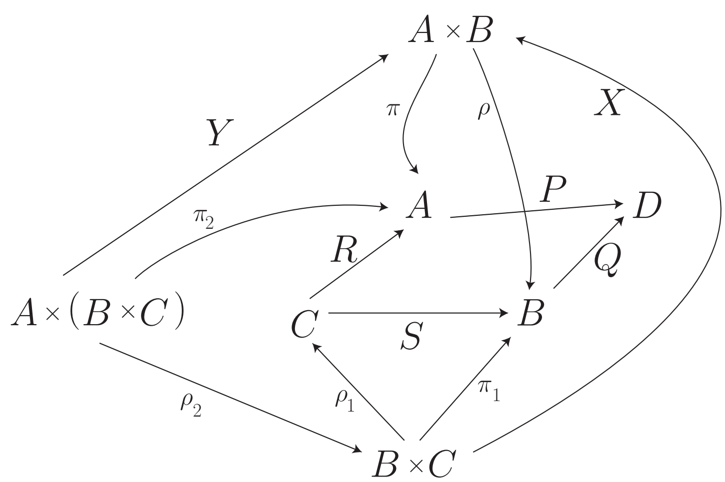

Fig. 7.2 The two additional products in the proof of $(R \olessthan S){;}(P \ogreaterthan Q) = R{;}P \cap S{;}Q$

For better reference, we recall an important result by Hans Zierer with its difficult proof from [Zie88, Zie91]. It shows that when these additional products are available, there will hold equality in the third containment of Prop. 7.3.i.

7.4 Proposition. Let again be given the setting above. When products π_1, ρ_1 and π_2, ρ_2,

$$\pi_1 : B \times C \longrightarrow B, \qquad \rho_1 : B \times C \longrightarrow C$$
$$\pi_2 : A \times (B \times C) \longrightarrow A, \qquad \rho_2 : A \times (B \times C) \longrightarrow (B \times C)$$

exist, there will in addition to Prop. 7.3.i hold

$$(R \olessthan S){;}(P \ogreaterthan Q) = R{;}P \cap S{;}Q.$$

Proof: The intricate point is to define the following constructs

$$X := \rho_1{;}R{;}\pi^\mathsf{T} \cap (\pi_1 \cap \rho_1{;}S){;}\rho^\mathsf{T} \qquad Y := (\pi_2 \cap \rho_2{;}\rho_1{;}R){;}\pi^\mathsf{T} \cap \rho_2{;}(\pi_1 \cap \rho_1{;}S){;}\rho^\mathsf{T},$$

of which Y turns out to be univalent, and to show several rather simple consequences. These follow applying the destroy and append-rule for univalent relations repeatedly.

$$\rho_1^\mathsf{T}{;}X = R{;}\pi^\mathsf{T} \cap S{;}\rho^\mathsf{T}$$
$$\rho_2^\mathsf{T}{;}Y = (\pi_1 \cap \rho_1{;}S){;}\rho^\mathsf{T} \cap \rho_1{;}R{;}\pi^\mathsf{T} = X$$
$$Y{;}\pi = (\pi_2 \cap \rho_2{;}\rho_1{;}R) \cap \rho_2{;}(\pi_1 \cap \rho_1{;}S){;}\mathbb{T} \qquad Y{;}\rho = (\pi_2 \cap \rho_2{;}\rho_1{;}R){;}\mathbb{T} \cap \rho_2{;}(\pi_1 \cap \rho_1{;}S)$$

Putting pieces together, we obtain

$$
\begin{aligned}
&(R \olessthan S){;}(P \ogreaterthan Q) = (R{;}\pi^\mathsf{T} \cap S{;}\rho^\mathsf{T}){;}(\pi{;}P \cap \rho{;}Q)\\
&= \rho_1^\mathsf{T}{;}\rho_2^\mathsf{T}{;}Y{;}(\pi{;}P \cap \rho{;}Q) \quad \text{see above}\\
&= \rho_1^\mathsf{T}{;}\rho_2^\mathsf{T}{;}(Y{;}\pi{;}P \cap Y{;}\rho{;}Q) \quad \text{since } Y \text{ is univalent}\\
&= \rho_1^\mathsf{T}{;}\rho_2^\mathsf{T}{;}\big[\{(\pi_2 \cap \rho_2{;}\rho_1{;}R) \cap \rho_2{;}(\pi_1 \cap \rho_1{;}S){;}\mathbb{T}\}{;}P\\
&\qquad\qquad \cap \{(\pi_2 \cap \rho_2{;}\rho_1{;}R){;}\mathbb{T} \cap \rho_2{;}(\pi_1 \cap \rho_1{;}S)\}{;}Q\big] \quad \text{see above}\\
&= \rho_1^\mathsf{T}{;}\rho_2^\mathsf{T}{;}\big[(\pi_2 \cap \rho_2{;}\rho_1{;}R){;}P \cap \rho_2{;}(\pi_1 \cap \rho_1{;}S){;}\mathbb{T}\\
&\qquad\qquad \cap (\pi_2 \cap \rho_2{;}\rho_1{;}R){;}\mathbb{T} \cap \rho_2{;}(\pi_1 \cap \rho_1{;}S){;}Q\big] \quad \text{masking}\\
&= \rho_1^\mathsf{T}{;}\rho_2^\mathsf{T}{;}\big[(\pi_2 \cap \rho_2{;}\rho_1{;}R){;}P \cap \rho_2{;}(\pi_1 \cap \rho_1{;}S){;}Q\big] \quad \text{trivial}\\
&= \rho_1^\mathsf{T}{;}\big[\rho_1{;}R{;}P \cap (\pi_1 \cap \rho_1{;}S){;}Q\big] \quad \text{destroy and append twice, } \rho_2^\mathsf{T}{;}\pi_2 = \mathbb{T}\\
&= R{;}P \cap S{;}Q \quad \text{destroy and append twice, } \rho_1^\mathsf{T}{;}\pi_1 = \mathbb{T}
\end{aligned}
$$

□

That the two products requested are often met in practice may be seen from the discussion of associativity in Def. 8.2 and Fig. 8.2.

It is the merit of Jules Desharnais, to have sharpened the previous result in the important paper [Des99]; also to be retrieved in [Win98]. Now just one of the relations P, Q, R, S needs to possess a vectorization in order to obtain equality.

7.5 Proposition. Let again be given the setting above. When the product π', ρ'

$$\pi' : A \times C \longrightarrow A, \qquad\qquad \rho' : A \times C \longrightarrow C$$

exists, there will in addition to Prop. 7.3.i hold

$$(R \olessthan S)\mathbin{;}(P \ogreaterthan Q) \;=\; R\mathbin{;}P \cap S\mathbin{;}Q.$$

Proof: Only one direction needs to be proved.

$$\begin{array}{ll}
R\mathbin{;}P \cap S\mathbin{;}Q = \;(R \olessthan \mathbb{I})\mathbin{;}\pi'\mathbin{;}P \cap S\mathbin{;}Q & \text{Prop. 7.2.iv} \\
\subseteq \big[(R \olessthan \mathbb{I}) \cap S\mathbin{;}Q\mathbin{;}P^{\mathsf{T}}\mathbin{;}\pi'^{\mathsf{T}}\big]\mathbin{;}\big[\pi'\mathbin{;}P \cap (R \olessthan \mathbb{I})^{\mathsf{T}}\mathbin{;}S\mathbin{;}Q\big] & \text{Dedekind rule} \\
\subseteq (R \olessthan \mathbb{I})\mathbin{;}\big[\pi'\mathbin{;}P \cap (R^{\mathsf{T}} \ogreaterthan \mathbb{I})\mathbin{;}S\mathbin{;}Q\big] & \text{monotony and transposition} \\
\subseteq (R \olessthan \mathbb{I})\mathbin{;}\big[\pi'\mathbin{;}P \cap \{(R^{\mathsf{T}} \ogreaterthan \mathbb{I})\mathbin{;}S\mathbin{;}Q \cap \pi'\mathbin{;}P\}\big] & \text{trivial} \\
\subseteq (R \olessthan \mathbb{I})\mathbin{;}(\pi'\mathbin{;}P \cap \big\{(R^{\mathsf{T}} \ogreaterthan \mathbb{I})\mathbin{;}S \cap \pi'\mathbin{;}P\mathbin{;}Q^{\mathsf{T}}\big\}\mathbin{;}Q) & \text{Dedekind rule and monotony} \\
\subseteq (R \olessthan \mathbb{I})\mathbin{;}(\pi'\mathbin{;}P \cap \big\{\pi'\mathbin{;}R^{\mathsf{T}}\mathbin{;}S \cap \rho'\mathbin{;}S \cap \pi'\mathbin{;}P\mathbin{;}Q^{\mathsf{T}}\big\}\mathbin{;}Q) & \\
\subseteq (R \olessthan \mathbb{I})\mathbin{;}(\pi'\mathbin{;}P \cap \big\{\pi'\mathbin{;}\mathbb{T} \cap \rho'\mathbin{;}S\big\}\mathbin{;}Q) & \text{trivial} \\
\subseteq (R \olessthan \mathbb{I})\mathbin{;}(\pi'\mathbin{;}P \cap (\mathbb{T} \ogreaterthan S)\mathbin{;}Q) & \text{definition of join} \\
\subseteq (R \olessthan \mathbb{I})\mathbin{;}(\pi'\mathbin{;}P \cap (\mathbb{I} \otimes S)\mathbin{;}\rho\mathbin{;}Q) & \text{Prop. 7.2.iv} \\
\subseteq (R \olessthan \mathbb{I})\mathbin{;}(\mathbb{I} \otimes S)\mathbin{;}(\rho\mathbin{;}Q \cap (\mathbb{I} \otimes S)^{\mathsf{T}}\mathbin{;}\pi'\mathbin{;}P) & \text{Dedekind rule and monotony} \\
\subseteq (R \olessthan \mathbb{I})\mathbin{;}(\mathbb{I} \otimes S)\mathbin{;}(\pi\mathbin{;}P \cap \rho\mathbin{;}Q) & \text{Prop. 7.2.i} \\
= (R \olessthan \mathbb{I})\mathbin{;}(\mathbb{I} \otimes S)\mathbin{;}(P \ogreaterthan Q) & \text{definition of join} \\
= (R \olessthan S)\mathbin{;}(P \ogreaterthan Q) & \text{according to [Sch11] Prop. 7.5}
\end{array}$$

□

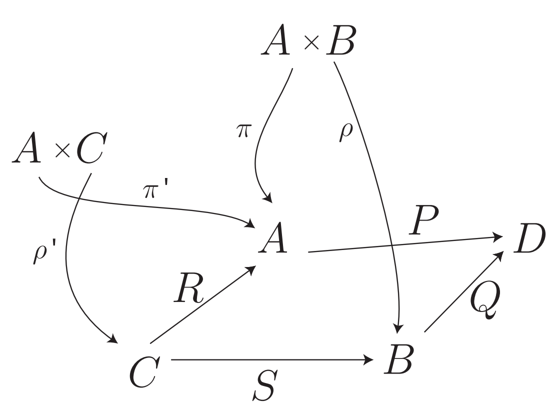

Fig. 7.3 The additional product in the proof of $(R \olessthan S)\mathbin{;}(P \ogreaterthan Q) = R\mathbin{;}P \cap S\mathbin{;}Q$

The following proposition states that left residuation distributes over the strict fork.

7.6 Proposition. For relations typed $A : W \longrightarrow X, \quad B : X \longrightarrow Y, \quad C : X \longrightarrow Z$

$$A\backslash (B \olessthan C) \;=\; (A\backslash B \olessthan A\backslash C)$$

Proof: $A\backslash\,(B \otimes C) \;=\; \overline{A^\top{;}\,\overline{(B \otimes C)}} = \overline{A^\top{;}\,\overline{B{;}\pi^\top \cap C{;}\rho^\top}} = \overline{A^\top{;}(\overline{B{;}\pi^\top} \cup \overline{C{;}\rho^\top})}$

$= \overline{A^\top{;}(\overline{B}{;}\pi^\top \cup \overline{C}{;}\rho^\top)} = \overline{A^\top{;}\overline{B}{;}\pi^\top \cup A^\top{;}\overline{C}{;}\rho^\top} = \overline{A^\top{;}\overline{B}{;}\pi^\top} \cap \overline{A^\top{;}\overline{C}{;}\rho^\top} = \overline{A^\top{;}\overline{B}}{;}\pi^\top \cap \overline{A^\top{;}\overline{C}}{;}\rho^\top$ □

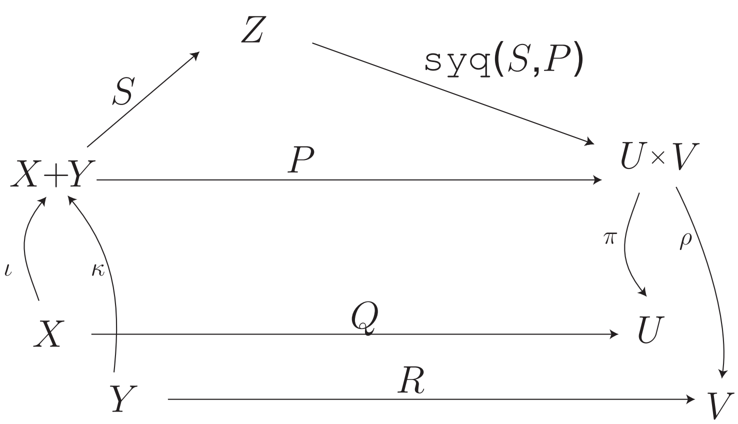

Fig. 7.4 Illustrating the addition theorem with $P := \iota^\top{;}Q{;}\pi^\top \cup \kappa^\top{;}R{;}\rho^\top$

An addition theorem, quite similar to the broadly known

$$\sin(x+y) = \sin x \cdot \cos y + \cos x \cdot \sin y,$$

holds for direct sum and direct product, cf. Fig. 7.4:

7.7 Proposition. Let be given any three relations $Q : X \longrightarrow U$, $R : Y \longrightarrow V$ and $S : X+Y \longrightarrow Z$. In addition, we consider the injections $\iota : X \longrightarrow X+Y$, $\kappa : Y \longrightarrow X+Y$ as well as the projections $\pi : U \times V \longrightarrow U$, $\rho : U \times V \longrightarrow V$, generically given. Then the following generalized addition theorem holds

$$\mathtt{syq}(S, \iota^\top{;}Q{;}\pi^\top \cup \kappa^\top{;}R{;}\rho^\top) = \mathtt{syq}(\iota{;}S, Q){;}\pi^\top \cap \mathtt{syq}(\kappa{;}S, R){;}\rho^\top.$$

In another notation, this looks as follows:

$$\mathtt{syq}(S, \iota^\top{;}Q{;}\pi^\top \cup \kappa^\top{;}R{;}\rho^\top) = \;(\mathtt{syq}(\iota{;}S, Q) \otimes \mathtt{syq}(\kappa{;}S, R))$$

Proof: In what follows, we abbreviate $P := \iota^\top{;}Q{;}\pi^\top \cup \kappa^\top{;}R{;}\rho^\top$.

$$\mathtt{syq}(\iota{;}S, Q){;}\pi^\top = \mathtt{syq}(\iota{;}S, Q{;}\pi^\top) = \mathtt{syq}(\iota{;}S, \iota{;}P)$$
$$\mathtt{syq}(\kappa{;}S, R){;}\rho^\top = \mathtt{syq}(\kappa{;}S, R{;}\rho^\top) = \mathtt{syq}(\kappa{;}S, \kappa{;}P)$$

$$\begin{aligned}
&\mathtt{syq}(\iota{;}S, \iota{;}P) \cap \mathtt{syq}(\kappa{;}S, \kappa{;}P) = \overline{\overline{S^\top{;}\iota^\top}{;}\iota{;}P} \cap \overline{S^\top{;}\iota^\top{;}\overline{\iota{;}P}} \cap \overline{\overline{S^\top{;}\kappa^\top}{;}\kappa{;}P} \cap \overline{S^\top{;}\kappa^\top{;}\overline{\kappa{;}P}}\\
&= \overline{\overline{S^\top}{;}\iota^\top{;}\iota{;}P} \cap \overline{S^\top{;}\iota^\top{;}\iota{;}\overline{P}} \cap \overline{\overline{S^\top}{;}\kappa^\top{;}\kappa{;}P} \cap \overline{S^\top{;}\kappa^\top{;}\kappa{;}\overline{P}} \quad \iota, \kappa \text{ are mappings}\\
&= \overline{\overline{S^\top}{;}\iota^\top{;}\iota{;}P} \cap \overline{\overline{S^\top}{;}\kappa^\top{;}\kappa{;}P} \cap \overline{S^\top{;}\iota^\top{;}\iota{;}\overline{P}} \cap \overline{S^\top{;}\kappa^\top{;}\kappa{;}\overline{P}} \quad \text{shuffled}\\
&= \overline{\overline{S^\top}{;}(\iota^\top{;}\iota \cup \kappa^\top{;}\kappa){;}P} \cap \overline{S^\top{;}(\iota^\top{;}\iota \cup \kappa^\top{;}\kappa){;}\overline{P}}\\
&= \overline{\overline{S^\top}{;}P} \cap \overline{S^\top{;}\overline{P}} = \mathtt{syq}(S, P) \quad \iota, \kappa \text{ form a direct sum}
\end{aligned}$$
□

Now we relate pairs of subsets of two sets X, Y with subsets of the direct sum $X+Y$.

$$\begin{tikzcd}
& 2^{X+Y} \arrow[loop above, "\Omega'"] \arrow[dr, "\varphi"] & \\
X+Y \arrow[ur, "\varepsilon'"] \arrow[rr, "\varepsilon_+"'] & & 2^X \times 2^Y \arrow[loop above, "\Omega_+"] \arrow[d, bend right, "\pi"'] \arrow[dd, bend left, "\rho"] \\
X \arrow[u, bend left, "\iota"] \arrow[rr, "\varepsilon_X"] & & 2^X \\
Y \arrow[uu, bend right, "\kappa"'] \arrow[rr, "\varepsilon_Y"] & & 2^Y
\end{tikzcd}$$

Fig. 7.5 Converting subsets of a sum to pairs of subsets

In Prop. 7.8, it is demonstrated that $\mathbf{2}^{X+Y}$ is isomorphic to $\mathbf{2}^X \times \mathbf{2}^Y$. In addition, it turns out that $\varepsilon_+ : X + Y \longrightarrow \mathbf{2}^X \times \mathbf{2}^Y$ satisfies the properties of a relational power, although it is constructed differently. In largely the same sense, the mappings

$$\mathtt{syq}(\iota;\varepsilon', \varepsilon_X) : \mathbf{2}^{X+Y} \longrightarrow \mathbf{2}^X \quad \text{and} \quad \mathtt{syq}(\kappa;\varepsilon', \varepsilon_Y) : \mathbf{2}^{X+Y} \longrightarrow \mathbf{2}^Y$$

establish $\mathbf{2}^{X+Y}$ as another direct product of $\mathbf{2}^X$ and $\mathbf{2}^Y$. Since the direct product is uniquely determined up to isomorphism, however, we are able to prove the isomorphism via the bijective mapping φ.

Earlier, we have been scrupulous with regard to the existence of products; we should maintain this here. Everything is fine, when ε', π, ρ are available. We assume this to be the case. In another model of relation algebra, however, π, ρ may not exist; then additional investigations are necessary.

7.8 Proposition. Let arbitrary sets X, Y be given for which we consider their membership relations $\varepsilon_X : X \longrightarrow \mathbf{2}^X$, $\varepsilon_Y : Y \longrightarrow \mathbf{2}^Y$, the direct product $\mathbf{2}^X \times \mathbf{2}^Y$ of these powersets, as well as the direct sum $X + Y$ and its membership relation $\varepsilon' : X + Y \longrightarrow \mathbf{2}^{X+Y}$; see Fig. 7.5. Then the following hold

i) for the construct $\quad \varepsilon_+ := \iota^\mathsf{T};\varepsilon_X;\pi^\mathsf{T} \cup \kappa^\mathsf{T};\varepsilon_Y;\rho^\mathsf{T}$

- $\iota;\varepsilon_+ = \varepsilon_X;\pi^\mathsf{T} \qquad \iota;\varepsilon_+;\pi = \varepsilon_X \qquad \kappa;\varepsilon_+ = \varepsilon_Y;\rho^\mathsf{T} \qquad \kappa;\varepsilon_+;\rho = \varepsilon_Y$
- $\mathtt{syq}(\iota;\varepsilon_+, \varepsilon_X) = \pi \qquad \mathtt{syq}(\kappa;\varepsilon_+, \varepsilon_Y) = \rho$

ii) for the construct $\quad \varphi := \mathtt{syq}(\varepsilon', \varepsilon_+)$

- $\varphi = \mathtt{syq}(\iota;\varepsilon', \varepsilon_X);\pi^\mathsf{T} \cap \mathtt{syq}(\kappa;\varepsilon', \varepsilon_Y);\rho^\mathsf{T}$, i.e. φ satisfies an addition theorem
- $\varphi;\pi = \mathtt{syq}(\iota;\varepsilon', \varepsilon_X) \qquad \varphi;\rho = \mathtt{syq}(\kappa;\varepsilon', \varepsilon_Y)$
- φ is a bijective mapping
- $\varepsilon';\varphi = \varepsilon_+ \qquad \varepsilon_+;\varphi^\mathsf{T} = \varepsilon'$
- $\mathtt{syq}(\pi;\varepsilon_X^\mathsf{T}, \varepsilon_+^\mathsf{T}) = \iota \qquad \mathtt{syq}(\rho;\varepsilon_Y^\mathsf{T}, \varepsilon_+^\mathsf{T}) = \kappa$
- ε_+ satisfies the relational requirements of a membership relation.

iii) for the construct $\Omega_+ := \overline{\overline{\varepsilon_+}^{\mathsf{T}};\varepsilon_+}$

- is an ordering.
- $\Omega_+ = (\Omega_X \otimes \Omega_Y)$
- φ is an order isomorphism between the orderings
  $\Omega' := \overline{\varepsilon'^{\mathsf{T}};\overline{\varepsilon'}} : \mathbf{2}^{X+Y} \longrightarrow \mathbf{2}^{X+Y}$ and $\Omega_+ : \mathbf{2}^X \times \mathbf{2}^Y \longrightarrow \mathbf{2}^X \times \mathbf{2}^Y$.

Proof: i) We demonstrate the main sample cases:

$$\iota;\varepsilon_+ = \iota;(\iota^{\mathsf{T}};\varepsilon_X;\pi^{\mathsf{T}} \cup \kappa^{\mathsf{T}};\varepsilon_Y;\rho^{\mathsf{T}}) = \iota;\iota^{\mathsf{T}};\varepsilon_X;\pi^{\mathsf{T}} \cup \iota;\kappa^{\mathsf{T}};\varepsilon_Y;\rho^{\mathsf{T}} = \mathbb{I};\varepsilon_X;\pi^{\mathsf{T}} \cup \perp\!\!\!\perp = \varepsilon_X;\pi^{\mathsf{T}}$$
$$\iota;\varepsilon_+;\pi = \varepsilon_X;\pi^{\mathsf{T}};\pi = \varepsilon_X$$
$$\mathtt{syq}(\iota;\varepsilon_+,\varepsilon_X) = \mathtt{syq}(\varepsilon_X;\pi^{\mathsf{T}},\varepsilon_X) = \pi;\mathtt{syq}(\varepsilon_X,\varepsilon_X) = \pi$$

ii) The first formula is an immediate consequence of the addition theorem Prop. 7.7. We have to obey some care: Only $\varepsilon_X, \varepsilon_Y, \varepsilon'$ have been introduced as membership relations; ε_+ is defined differently but denoted similarly, since it will soon turn out to be one also.

Then we prove with the addition theorem

$$\begin{aligned}
&\varphi;\pi = \left[\mathtt{syq}(\iota;\varepsilon',\varepsilon_X);\pi^{\mathsf{T}} \cap \mathtt{syq}(\kappa;\varepsilon',\varepsilon_Y);\rho^{\mathsf{T}}\right];\pi \\
&= \mathtt{syq}(\iota;\varepsilon',\varepsilon_X) \cap \mathtt{syq}(\kappa;\varepsilon',\varepsilon_Y);\rho^{\mathsf{T}};\pi \\
&= \mathtt{syq}(\iota;\varepsilon',\varepsilon_X) \cap \mathtt{syq}(\kappa;\varepsilon',\varepsilon_Y);\mathbb{T} \qquad \text{since } \pi,\rho \text{ form a direct product} \\
&= \mathtt{syq}(\iota;\varepsilon',\varepsilon_X) \cap \mathbb{T} \qquad \text{since } \mathtt{syq}(\varepsilon_Y,\ldots) \text{ is always surjective} \\
&= \mathtt{syq}(\iota;\varepsilon',\varepsilon_X)
\end{aligned}$$

Now, we convince ourselves that φ is total, which follows with the preceding result from

$$\varphi;\mathbb{T} = \varphi;\pi;\mathbb{T} = \mathtt{syq}(\iota;\varepsilon',\varepsilon_X);\mathbb{T} \quad \text{and the fact that } \varepsilon_X \text{ is a membership}$$

Univalency follows also with the addition theorem

$$\begin{aligned}
&\varphi^{\mathsf{T}};\varphi \subseteq \left[\pi;\mathtt{syq}(\varepsilon_X,\iota;\varepsilon') \cap \rho;\mathtt{syq}(\varepsilon_Y,\kappa;\varepsilon')\right];\left[\mathtt{syq}(\iota;\varepsilon',\varepsilon_X);\pi^{\mathsf{T}} \cap \mathtt{syq}(\kappa;\varepsilon',\varepsilon_Y);\rho^{\mathsf{T}}\right] \\
&\subseteq \pi;\mathtt{syq}(\varepsilon_X,\iota;\varepsilon');\mathtt{syq}(\iota;\varepsilon',\varepsilon_X);\pi^{\mathsf{T}} \cap \rho;\mathtt{syq}(\varepsilon_Y,\kappa;\varepsilon');\mathtt{syq}(\kappa;\varepsilon',\varepsilon_Y);\rho^{\mathsf{T}} \\
&\subseteq \pi;\mathtt{syq}(\varepsilon_X,\varepsilon_X);\pi^{\mathsf{T}} \cap \rho;\mathtt{syq}(\varepsilon_Y,\varepsilon_Y);\rho^{\mathsf{T}} \subseteq \pi;\pi^{\mathsf{T}} \cap \rho;\rho^{\mathsf{T}} = \mathbb{I}
\end{aligned}$$

Even simpler and without the addition theorem we get $\varphi;\varphi^{\mathsf{T}} \subseteq \mathtt{syq}(\varepsilon',\varepsilon') \subseteq \mathbb{I}$, so that φ is injective. Finally, φ is surjective since ε' is a membership relation.

$$\begin{aligned}
&\varepsilon';\varphi = \varepsilon';\mathtt{syq}(\varepsilon',\varepsilon_+) = \varepsilon_+ \quad \text{since } \varepsilon' \text{ is a membership relation} \\
&\varepsilon_+;\varphi^{\mathsf{T}} = \varepsilon';\varphi;\varphi^{\mathsf{T}} = \varepsilon' \qquad \text{since } \varphi \text{ is already established as a bijective mapping}
\end{aligned}$$

$$\begin{aligned}
&\mathtt{syq}(\pi;\varepsilon_X^{\mathsf{T}},\varepsilon_+^{\mathsf{T}}) = \mathtt{syq}(\pi;\varepsilon_X^{\mathsf{T}},\varphi^{\mathsf{T}};\varepsilon'^{\mathsf{T}}) = \mathtt{syq}(\varphi;\pi;\varepsilon_X^{\mathsf{T}},\varepsilon'^{\mathsf{T}}) \quad \text{since } \varphi \text{ is a bijective mapping} \\
&= \mathtt{syq}(\mathtt{syq}(\iota;\varepsilon',\varepsilon_X);\varepsilon_X^{\mathsf{T}},\varepsilon'^{\mathsf{T}}) \quad \text{see above} \\
&= \mathtt{syq}(\varepsilon'^{\mathsf{T}};\iota^{\mathsf{T}},\varepsilon'^{\mathsf{T}}) \\
&= \iota;\mathtt{syq}(\varepsilon'^{\mathsf{T}},\varepsilon'^{\mathsf{T}}) \\
&= \iota;(\overline{\overline{\varepsilon'};\varepsilon'^{\mathsf{T}}} \cap \overline{\varepsilon';\overline{\varepsilon'^{\mathsf{T}}}}) \\
&= \iota;(\Omega'^{\mathsf{T}} \cap \Omega') = \iota;\mathbb{I} = \iota
\end{aligned}$$

It is relatively easy to prove that the differently constructed ε_+ is a membership relation:

$\mathtt{syq}(\varepsilon_+, \varepsilon_+) = \mathtt{syq}(\varepsilon';\varphi, \varepsilon';\varphi) = \varphi^\mathsf{T};\mathtt{syq}(\varepsilon', \varepsilon';\varphi) = \varphi^\mathsf{T};\mathtt{syq}(\varepsilon', \varepsilon');\varphi = \varphi^\mathsf{T};\mathbb{I};\varphi = \varphi^\mathsf{T};\varphi = \mathbb{I}$

$\mathtt{syq}(\varepsilon_+, U) = \mathtt{syq}(\varepsilon';\varphi, U) = \varphi^\mathsf{T};\mathtt{syq}(\varepsilon', U)$ is surjective since ε' is a membership according to [Sch11] Prop. 8.18.

iii) Ω_+ is — consequently — indeed an ordering. It satisfies

$$\begin{aligned}
(\Omega_X \bigotimes \Omega_Y) &= \pi;\overline{\varepsilon_X^\mathsf{T};\overline{\varepsilon_X}};\pi^\mathsf{T} \cap \rho;\overline{\varepsilon_Y^\mathsf{T};\overline{\varepsilon_Y}};\rho^\mathsf{T} = \overline{\pi;\varepsilon_X^\mathsf{T};\overline{\varepsilon_X;\pi^\mathsf{T}}} \cap \overline{\rho;\varepsilon_Y^\mathsf{T};\overline{\varepsilon_Y;\rho^\mathsf{T}}} \\
&= \overline{\varepsilon_+^\mathsf{T};\iota^\mathsf{T};\overline{\iota;\varepsilon_+}} \cap \overline{\varepsilon_+^\mathsf{T};\kappa^\mathsf{T};\overline{\kappa;\varepsilon_+}} \quad \text{Prop. 7.8.i} \\
&= \overline{\varepsilon_+^\mathsf{T};\iota^\mathsf{T};\iota;\overline{\varepsilon_+}} \cap \overline{\varepsilon_+^\mathsf{T};\kappa^\mathsf{T};\kappa;\overline{\varepsilon_+}} \quad \iota, \kappa \text{ are mappings} \\
&= \overline{\varepsilon_+^\mathsf{T};\iota^\mathsf{T};\iota;\overline{\varepsilon_+} \cup \varepsilon_+^\mathsf{T};\kappa^\mathsf{T};\kappa;\overline{\varepsilon_+}} \\
&= \overline{\varepsilon_+^\mathsf{T};(\iota^\mathsf{T};\iota \cup \kappa^\mathsf{T};\kappa);\overline{\varepsilon_+}} = \overline{\varepsilon_+^\mathsf{T};\overline{\varepsilon_+}} \quad \iota, \kappa \text{ form a direct sum} \\
&= \Omega_+
\end{aligned}$$

First direction of the isomorphism proposition, using that φ is a bijective mapping:

$$\Omega';\varphi = \overline{\varepsilon'^\mathsf{T};\overline{\varepsilon'}};\varphi = \overline{\varepsilon'^\mathsf{T};\overline{\varepsilon';\varphi}} = \overline{\varepsilon'^\mathsf{T};\overline{\varepsilon_+}} = \overline{\varphi;\varepsilon_+^\mathsf{T};\overline{\varepsilon_+}} = \varphi;\overline{\varepsilon_+^\mathsf{T};\overline{\varepsilon_+}} = \varphi;\Omega_+$$

Second direction:

$$\Omega_+;\varphi^\mathsf{T} = \overline{\varepsilon_+^\mathsf{T};\overline{\varepsilon_+}};\varphi^\mathsf{T} = \overline{\varepsilon_+^\mathsf{T};\overline{\varepsilon_+;\varphi^\mathsf{T}}} = \overline{\varepsilon_+^\mathsf{T};\overline{\varepsilon'}} = \overline{\varphi^\mathsf{T};\varepsilon'^\mathsf{T};\overline{\varepsilon'}} = \varphi^\mathsf{T};\overline{\varepsilon'^\mathsf{T};\overline{\varepsilon'}} = \varphi^\mathsf{T};\Omega' \qquad \square$$

8 Binary operations

We now attempt to study also binary operations on a set relationally. This will already allow a very basic look on group theory. It will turn out that such elements as the unit, e.g., will be points. A *point* resembles the classic *element* of set theory. In the relational setting, a *point* is a row-constant, injective, and surjective relation x, i.e, it satisfies

$$x;\mathbb{T} = x, \qquad x;x^\mathsf{T} \subseteq \mathbb{I}, \qquad \mathbb{T};x = \mathbb{T}.$$

We assume a direct product with projections $\pi, \rho : X \times X \longrightarrow X$ and in addition a binary mapping $\mathfrak{A} : X \times X \longrightarrow X$. A first preparatory observation concerns what one might consider as coretract or section in a category, here simply a left-inverse of the projection ρ.

8.1 Proposition. If x is any point, then $f := (\rho \cap \pi;x;\mathbb{T})^\mathsf{T}$ is a mapping. It satisfies $f;\rho = \mathbb{I}$ and $\rho \subseteq f\backslash\mathbb{I}$.

Proof: Since x is row-constant and injective, we have univalence

$$f^\mathsf{T};f = (\rho \cap \pi;x;\mathbb{T});(\rho \cap \pi;x;\mathbb{T})^\mathsf{T} \subseteq \rho;\rho^\mathsf{T} \cap \pi;x;\mathbb{T};\mathbb{T}^\mathsf{T};x^\mathsf{T};\pi^\mathsf{T} \subseteq \rho;\rho^\mathsf{T} \cap \pi;\pi^\mathsf{T} = \mathbb{I}$$

as well as totality

$$f;\mathbb{T} = (\rho^\mathsf{T} \cap \mathbb{T}^\mathsf{T};x^\mathsf{T};\pi^\mathsf{T});\mathbb{T} = \rho^\mathsf{T};(\mathbb{T} \cap \pi;x;\mathbb{T}) = \rho^\mathsf{T};\pi;x;\mathbb{T} = \mathbb{T};x;\mathbb{T} = \mathbb{T} \quad \text{since } x \text{ is a point.}$$

$$f;\rho = (\rho^\mathsf{T} \cap \mathbb{T};x^\mathsf{T};\pi^\mathsf{T});\rho = \mathbb{I} \cap \mathbb{T};x^\mathsf{T};\pi^\mathsf{T};\rho = \mathbb{I} \cap \mathbb{T};x^\mathsf{T};\mathbb{T} = \mathbb{I} \cap \mathbb{T} = \mathbb{I}$$

$$\rho \subseteq f\backslash\mathbb{I} = \overline{f^\mathsf{T};\overline{\mathbb{I}}} \iff f^\mathsf{T};\overline{\mathbb{I}} \subseteq \overline{\rho} \iff f;\rho \subseteq \mathbb{I} \iff \mathtt{True} \qquad \square$$

$$\begin{array}{c@{}c} & \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ g \end{array} & \begin{pmatrix} 3 & 2 & 1 & 4 & 1 & 6 & 7 \\ 1 & 2 & 5 & 4 & 3 & 6 & 7 \\ 1 & 3 & 2 & 4 & 2 & 6 & 7 \\ 1 & 5 & 4 & 7 & 4 & 3 & 6 \\ 1 & 2 & 5 & 4 & 5 & 6 & 7 \\ 1 & 3 & 6 & 5 & 2 & 4 & 7 \\ 1 & 2 & 7 & 4 & 7 & 6 & 3 \end{pmatrix} \end{array}$$

number indicates row

$$\text{point } c = \begin{array}{c@{}c} \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ g \end{array} & \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{array}$$

table $\mathfrak{A}$

$$\begin{array}{c@{}c} & \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{array}{c} (a,a) \\ (b,a) \\ (a,b) \\ (c,a) \\ (b,b) \\ (a,c) \\ (d,a) \\ (c,b) \\ (b,c) \\ (a,d) \\ (e,a) \\ (d,b) \\ (c,c) \\ (b,d) \\ (a,e) \\ (f,a) \\ (e,b) \\ (d,c) \\ (c,d) \\ (b,e) \\ (a,f) \\ (g,a) \\ (f,b) \\ (e,c) \\ (d,d) \\ (c,e) \\ (b,f) \\ (a,g) \\ (g,b) \\ (f,c) \\ (e,d) \\ (d,e) \\ (c,f) \\ (b,g) \\ (g,c) \\ (f,d) \\ (e,e) \\ (d,f) \\ (c,g) \\ (g,d) \\ (f,e) \\ (e,f) \\ (d,g) \\ (g,e) \\ (f,f) \\ (e,g) \\ (g,f) \\ (f,g) \\ (g,g) \end{array} & \begin{pmatrix}
0&0&1&0&0&0&0\\
1&0&0&0&0&0&0\\
0&1&0&0&0&0&0\\
1&0&0&0&0&0&0\\
0&0&1&0&0&0&0\\
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0&0&0&0&0&1&0\\
0&0&0&0&0&0&1\\
0&0&0&1&0&0&0\\
0&0&0&0&0&0&1\\
0&0&0&0&0&1&0\\
0&0&0&0&0&0&1\\
0&0&1&0&0&0&0
\end{pmatrix} \end{array}$$

relation $\mathfrak{A}$

$$\begin{array}{c@{}c} & \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{array}{c} (a,a) \\ (b,a) \\ (a,b) \\ (c,a) \\ (b,b) \\ (a,c) \\ (d,a) \\ (c,b) \\ (b,c) \\ (a,d) \\ (e,a) \\ (d,b) \\ (c,c) \\ (b,d) \\ (a,e) \\ (f,a) \\ (e,b) \\ (d,c) \\ (c,d) \\ (b,e) \\ (a,f) \\ (g,a) \\ (f,b) \\ (e,c) \\ (d,d) \\ (c,e) \\ (b,f) \\ (a,g) \\ (g,b) \\ (f,c) \\ (e,d) \\ (d,e) \\ (c,f) \\ (b,g) \\ (g,c) \\ (f,d) \\ (e,e) \\ (d,f) \\ (c,g) \\ (g,d) \\ (f,e) \\ (e,f) \\ (d,g) \\ (g,e) \\ (f,f) \\ (e,g) \\ (g,f) \\ (f,g) \\ (g,g) \end{array} & \begin{pmatrix}
1&0&0&0&0&0&0\\
1&0&0&0&0&0&0\\
0&1&0&0&0&0&0\\
1&0&0&0&0&0&0\\
0&1&0&0&0&0&0\\
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0&0&0&0&0&1&0\\
0&0&0&0&0&0&1\\
0&0&0&0&0&1&0\\
0&0&0&0&0&0&1\\
0&0&0&0&0&0&1
\end{pmatrix} \end{array}$$

ρ

$$\begin{array}{c@{}c} & \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{array}{c} (a,a) \\ (b,a) \\ (a,b) \\ (c,a) \\ (b,b) \\ (a,c) \\ (d,a) \\ (c,b) \\ (b,c) \\ (a,d) \\ (e,a) \\ (d,b) \\ (c,c) \\ (b,d) \\ (a,e) \\ (f,a) \\ (e,b) \\ (d,c) \\ (c,d) \\ (b,e) \\ (a,f) \\ (g,a) \\ (f,b) \\ (e,c) \\ (d,d) \\ (c,e) \\ (b,f) \\ (a,g) \\ (g,b) \\ (f,c) \\ (e,d) \\ (d,e) \\ (c,f) \\ (b,g) \\ (g,c) \\ (f,d) \\ (e,e) \\ (d,f) \\ (c,g) \\ (g,d) \\ (f,e) \\ (e,f) \\ (d,g) \\ (g,e) \\ (f,f) \\ (e,g) \\ (g,f) \\ (f,g) \\ (g,g) \end{array} & \begin{pmatrix}
0&0&0&0&0&0&0\\
0&0&0&0&0&0&0\\
0&0&0&0&0&0&0\\
1&0&0&0&0&0&0\\
0&0&0&0&0&0&0\\
0&0&0&0&0&0&0\\
0&0&0&0&0&0&0\\
0&1&0&0&0&0&0\\
0&0&0&0&0&0&0\\
0&0&0&0&0&0&0\\
0&0&0&0&0&0&0\\
0&0&0&0&0&0&0\\
0&0&1&0&0&0&0\\
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0&0&0&0&0&0&0\\
0&0&0&0&0&0&0\\
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0&0&0&0&1&0&0\\
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0&0&0&0&0&0&0\\
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0&0&0&0&0&0&0\\
0&0&0&0&0&0&0\\
0&0&0&0&0&0&0\\
0&0&0&0&0&1&0\\
0&0&0&0&0&0&0\\
0&0&0&0&0&0&0\\
0&0&0&0&0&0&0\\
0&0&0&0&0&0&0\\
0&0&0&0&0&0&0\\
0&0&0&0&0&0&1\\
0&0&0&0&0&0&0\\
0&0&0&0&0&0&0\\
0&0&0&0&0&0&0\\
0&0&0&0&0&0&0\\
0&0&0&0&0&0&0\\
0&0&0&0&0&0&0\\
0&0&0&0&0&0&0\\
0&0&0&0&0&0&0\\
0&0&0&0&0&0&0\\
0&0&0&0&0&0&0
\end{pmatrix} \end{array}$$

f^T

Fig. 8.1 Binary map as table and as relation, projection ρ and transposed mapping f for point c

Given x, we map with f every y to the pair (x, y). By symmetry, $g := (\pi \cap \rho{:}x{:}\mathbb{T})^\mathsf{T}$ is also a mapping; it satisfies $g{:}\pi = \mathbb{I}$ and $\pi = g\backslash\mathbb{I}$.

8.2 Definition. Given this setting, we define as follows:

i) $P := \pi{:}\rho^\mathsf{T} \cap \rho{:}\pi^\mathsf{T}$ flips components of a pair.

ii) $\mathfrak{A}$ **commutative** $:\Longleftrightarrow \quad P;\mathfrak{A} = \mathfrak{A}$.

iii) The shuffling for the associative law is achieved by one version of the following

$$T := \pi';\pi;\pi_1^{\mathsf{T}} \cap \pi';\rho;\pi^{\mathsf{T}};\rho_1^{\mathsf{T}} \cap \rho';\rho^{\mathsf{T}};\rho_1^{\mathsf{T}} \quad \text{or, grouped suitably,}$$

$$= \pi';\pi;\pi_1^{\mathsf{T}} \cap (\pi';\rho;\pi^{\mathsf{T}} \cap \rho';\rho^{\mathsf{T}});\rho_1^{\mathsf{T}} = (\pi';\pi \olessthan (\rho \otimes \mathbb{I})\,)$$

$$= \pi';(\pi;\pi_1^{\mathsf{T}} \cap \rho;\pi^{\mathsf{T}};\rho_1^{\mathsf{T}}) \cap \rho';\rho^{\mathsf{T}};\rho_1^{\mathsf{T}} = (\,(\mathbb{I} \otimes \pi^{\mathsf{T}}) \ogreaterthan \rho^{\mathsf{T}};\rho_1^{\mathsf{T}})$$

iv) $\mathfrak{A}$ **associative** $:\Longleftrightarrow \quad (\mathfrak{A} \otimes \mathbb{I}_X);\mathfrak{A} = T;(\mathbb{I}_X \otimes \mathfrak{A});\mathfrak{A}$ □

The associativity condition is here given in an acceptably concise form; written down without sufficient care, it appears considerably longer.

8.3 Lemma. Several identities for P, T — correct typing assumed.

i) P, T are bijective mappings.

ii) $P^{\mathsf{T}} = P$

iii) $P;(R \otimes S) = (S \otimes R);P'$

iv) $P;(R \ogreaterthan S) = (S \ogreaterthan R) \qquad (R \olessthan S);P' = (S \olessthan R)$

v) $T;(Q \otimes (R \otimes S)\,) = (\,(Q \otimes R) \otimes S);T'$

vi) $(Q \olessthan (R \olessthan S)\,) = (\,(Q \olessthan R) \olessthan S);T'$

Proof: iii) $P;(R \otimes S) = (\pi;\rho^{\mathsf{T}} \cap \rho;\pi^{\mathsf{T}});(\pi;R;\pi'^{\mathsf{T}} \cap \rho;S;\rho'^{\mathsf{T}})$ by definition
$= (\pi;\rho^{\mathsf{T}} \cap \rho;\pi^{\mathsf{T}});\pi;R;\pi'^{\mathsf{T}} \cap (\pi;\rho^{\mathsf{T}} \cap \rho;\pi^{\mathsf{T}});\rho;S;\rho'^{\mathsf{T}}$ since P is univalent
$= (\pi;\rho^{\mathsf{T}};\pi \cap \rho);R;\pi'^{\mathsf{T}} \cap (\pi \cap \rho;\pi^{\mathsf{T}};\rho);S;\rho'^{\mathsf{T}}$ Prop. 7.2.ii
$= \rho;R;\pi'^{\mathsf{T}} \cap \pi;S;\rho'^{\mathsf{T}} = \pi;S;\rho'^{\mathsf{T}} \cap \rho;R;\pi'^{\mathsf{T}}$ π, ρ are projections

Similarly from the other side:
$(S \otimes R);P'$
$= (\pi;S;\pi'^{\mathsf{T}} \cap \rho;R;\rho'^{\mathsf{T}});(\pi';\rho'^{\mathsf{T}} \cap \rho';\pi'^{\mathsf{T}})$
$= \pi;S;\pi'^{\mathsf{T}};(\pi';\rho'^{\mathsf{T}} \cap \rho';\pi'^{\mathsf{T}}) \cap \rho;R;\rho'^{\mathsf{T}};(\pi';\rho'^{\mathsf{T}} \cap \rho';\pi'^{\mathsf{T}})$
$= \pi;S;(\rho'^{\mathsf{T}} \cap \pi'^{\mathsf{T}};\rho';\pi'^{\mathsf{T}}) \cap \rho;R;(\rho'^{\mathsf{T}};\pi';\rho'^{\mathsf{T}} \cap \pi'^{\mathsf{T}})$
$= \pi;S;\rho'^{\mathsf{T}} \cap \rho;R;\pi'^{\mathsf{T}}$

iv) $P;(R \ogreaterthan S) = (\pi;\rho^{\mathsf{T}} \cap \rho;\pi^{\mathsf{T}});(\pi;R \cap \rho;S)$ by definition
$= (\pi;\rho^{\mathsf{T}} \cap \rho;\pi^{\mathsf{T}});\pi;R \cap (\pi;\rho^{\mathsf{T}} \cap \rho;\pi^{\mathsf{T}});\rho;S$ since P is univalent
$= (\pi;\rho^{\mathsf{T}};\pi \cap \rho);R \cap (\pi \cap \rho;\pi^{\mathsf{T}};\rho);S$ Prop. 7.2.ii
$= \rho;R \cap \pi;S = \pi;S \cap \rho;R$ π, ρ are projections

$$
\begin{tikzcd}
(X\times X)\times X \arrow[rr, "T"] \arrow[dr, "\pi'"'] \arrow[ddr, bend left, "\rho'"] & & X\times(X\times X) \arrow[dl, bend left, "\rho_1"] \arrow[ddl, bend left, "\pi_1"] \\
& X\times X \arrow[d, bend right, "\pi"'] \arrow[d, "\mathcal{A}"] \arrow[d, bend left, "\rho"] & \\
& X &
\end{tikzcd}
$$

$$
T = \begin{array}{rc}
 & \begin{array}{ccccccccccccccccccccccccccc} (1,(1,1)) & (2,(1,1)) & (1,(2,1)) & (3,(1,1)) & (2,(2,1)) & (1,(1,2)) & (3,(2,1)) & (2,(1,2)) & (1,(3,1)) & (3,(1,2)) & (2,(3,1)) & (1,(2,2)) & (3,(3,1)) & (2,(2,2)) & (1,(1,3)) & (3,(2,2)) & (2,(1,3)) & (1,(3,2)) & (3,(1,3)) & (2,(3,2)) & (1,(2,3)) & (3,(3,2)) & (2,(2,3)) & (1,(3,3)) & (3,(2,3)) & (2,(3,3)) & (3,(3,3)) \end{array} \\
\begin{array}{r} ((1,1),1) \\ ((2,1),1) \\ ((1,1),2) \\ ((1,2),1) \\ ((2,1),2) \\ ((1,1),3) \\ ((3,1),1) \\ ((1,2),2) \\ ((2,1),3) \\ ((2,2),1) \\ ((3,1),2) \\ ((1,2),3) \\ ((1,3),1) \\ ((2,2),2) \\ ((3,1),3) \\ ((3,2),1) \\ ((1,3),2) \\ ((2,2),3) \\ ((2,3),1) \\ ((3,2),2) \\ ((1,3),3) \\ ((3,3),1) \\ ((2,3),2) \\ ((3,2),3) \\ ((3,3),2) \\ ((2,3),3) \\ ((3,3),3) \end{array} &
\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}
\end{array}
$$

Fig. 8.2 Illustrating the associative shuffling

v) similar to (iii) and (vi)

vi) $((Q \olessthan R) \olessthan S);T = ((Q \olessthan R) \olessthan S);((\mathbb{I} \otimes \pi^{\mathsf{T}}) \ogreaterthan \rho^{\mathsf{T}};\rho_1^{\mathsf{T}})$ by definition of T

$= (Q \olessthan R);(\mathbb{I} \otimes \pi^{\mathsf{T}}) \cap S;\rho^{\mathsf{T}};\rho_1^{\mathsf{T}}$ Prop. 7.3.iii since $(\mathbb{I} \otimes \pi^{\mathsf{T}})$ and $\rho^{\mathsf{T}};\rho_1^{\mathsf{T}}$ are both injective

$= (Q;\mathbb{I} \olessthan R;\pi^{\mathsf{T}}) \cap S;\rho^{\mathsf{T}};\rho_1^{\mathsf{T}}$

$= Q;\pi_1^{\mathsf{T}} \cap R;\pi^{\mathsf{T}};\rho_1^{\mathsf{T}} \cap S;\rho^{\mathsf{T}};\rho_1^{\mathsf{T}}$

$= Q;\pi_1^{\mathsf{T}} \cap (R;\pi^{\mathsf{T}} \cap S;\rho^{\mathsf{T}});\rho_1^{\mathsf{T}}$

$= Q;\pi_1^{\mathsf{T}} \cap (R \olessthan S);\rho_1^{\mathsf{T}}$

$= (Q \olessthan (R \olessthan S))$ $\square$

Several identities are satisfied for T:

$$T;\pi_1 = \pi';\pi, \qquad\qquad T;\rho_1 = \pi';\rho;\pi^\mathsf{T} \cap \rho';\rho^\mathsf{T} = (\rho \otimes \mathbb{I})$$
$$\pi'^\mathsf{T};T = \pi;\pi_1^\mathsf{T} \cap \rho;\pi^\mathsf{T};\rho_1^\mathsf{T} = (\mathbb{I} \otimes \pi^\mathsf{T}), \qquad \rho'^\mathsf{T};T = \rho^\mathsf{T};\rho_1^\mathsf{T}$$

There follow characterizations of elements as being neutral, being inverses, etc. One will observe in (i), that the possibility of left-inversion of x, (i.e. $\forall y : \exists p : \pi_{px} \wedge \mathfrak{A}_{py}, \quad \forall y : \exists z : x + z = y$) is defined without mentioning the neutral element. A left-invertible element is characterized by the fact that the corresponding row of the composition table for $\mathfrak{A}$ contains all the elements in some sequence.

8.4 Definition. Let be given the binary mapping $\mathfrak{A}$ as before.

i) $\overline{\overline{\pi^\mathsf{T};\mathfrak{A};\mathbb{T}}}$ the set of elements that may be **left-inverted**, i.e., $\{x \mid \forall y : \exists p : \pi_{px} \wedge \mathfrak{A}_{py}\}$

ii) $\overline{\overline{\rho^\mathsf{T};\mathfrak{A};\mathbb{T}}}$ the set of elements that may be **right-inverted**, i.e., $\{y \mid \forall x : \exists p : \rho_{py} \wedge \mathfrak{A}_{px}\}$

iii) $\mathfrak{A}$ allows **left-inversion** $:\Longleftrightarrow \quad \pi^\mathsf{T};\mathfrak{A} = \mathbb{T}$

iv) $\mathfrak{A}$ allows **right-inversion** $:\Longleftrightarrow \quad \rho^\mathsf{T};\mathfrak{A} = \mathbb{T}$ □

To identify a left-invertible point e (i.e. a transposed map) means via shunting also

$$e \subseteq \overline{\overline{\pi^\mathsf{T};\mathfrak{A};\mathbb{T}}} \iff \mathbb{T} \subseteq \mathfrak{A}^\mathsf{T};\pi;e \iff e;\mathbb{T} \subseteq \pi^\mathsf{T};\mathfrak{A},$$

and relates (i) with (iii). In Fig. 8.3, for the element a, e.g., there is no element x such that $\mathfrak{A}_{ax} = e$, the fifth.

$$\mathfrak{A} = \begin{array}{c} \\ \mathrm{a}\\ \mathrm{b}\\ \mathrm{c}\\ \mathrm{d}\\ \mathrm{e}\\ \mathrm{f}\\ \mathrm{g}\end{array}\begin{array}{c}\begin{array}{ccccccc}\mathrm{a}&\mathrm{b}&\mathrm{c}&\mathrm{d}&\mathrm{e}&\mathrm{f}&\mathrm{g}\end{array}\\ \begin{pmatrix} 3&2&1&4&1&6&7\\ 1&2&5&4&3&6&7\\ 1&3&2&4&2&6&7\\ 1&5&4&7&4&3&6\\ 1&2&5&4&5&6&7\\ 1&3&6&5&2&4&7\\ 1&2&7&4&7&6&3 \end{pmatrix}\end{array} \qquad \overline{\overline{\pi^\mathsf{T};\mathfrak{A};\mathbb{T}}} = \begin{array}{c}\mathrm{a}\\ \mathrm{b}\\ \mathrm{c}\\ \mathrm{d}\\ \mathrm{e}\\ \mathrm{f}\\ \mathrm{g}\end{array}\begin{pmatrix}0\\1\\0\\0\\0\\1\\0\end{pmatrix} \qquad \mathfrak{A}^\mathsf{T};(\rho \cap \pi;f;\mathbb{T}) = \begin{array}{c} \\ \mathrm{a}\\ \mathrm{b}\\ \mathrm{c}\\ \mathrm{d}\\ \mathrm{e}\\ \mathrm{f}\\ \mathrm{g}\end{array}\begin{array}{c}\begin{array}{ccccccc}\mathrm{a}&\mathrm{b}&\mathrm{c}&\mathrm{d}&\mathrm{e}&\mathrm{f}&\mathrm{g}\end{array}\\ \begin{pmatrix} 1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&1&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix}\end{array}$$

Fig. 8.3 A mapping with elements b, f posessing all left-inverses and left-division by f

Whenever one takes a point $i \subseteq \overline{\overline{\pi^\mathsf{T};\mathfrak{A};\mathbb{T}}}$, the construct $f := (\rho \cap \pi;i;\mathbb{T})^\mathsf{T}$ is a mapping, according to Prop. 8.1. As an example, left-division by f is shown as a mapping $\mathfrak{A}^\mathsf{T};(\rho \cap \pi;f;\mathbb{T})$ on the right: $\quad \mathfrak{A}_{\mathrm{f}\,\mathrm{b}} = c \implies \mathrm{f}\backslash\mathrm{c} = \mathrm{b}$ or else $\mathfrak{A}_{\mathrm{f}\,\mathrm{d}} = \mathrm{e} \implies \mathrm{f}\backslash\mathrm{e} = \mathrm{d}$.

Invariant elements commute with every other one. In the table representation, row and column concerning this element are equal.

8.5 Definition. Let be given the binary mapping $\mathfrak{A}$ as before. Then

$\overline{\pi^{\mathsf{T}};\overline{[\mathfrak{A}\cap P;\mathfrak{A}];\mathbb{T}}} = \overline{\rho^{\mathsf{T}};\overline{[\mathfrak{A}\cap P;\mathfrak{A}];\mathbb{T}}} = \pi\backslash([\mathfrak{A}\cap P;\mathfrak{A}];\mathbb{T})$ is the set of **invariant elements**, i.e., those x with $\forall y : \mathfrak{A}_{xy} = \mathfrak{A}_{yx}$ □

$$\mathfrak{A} = \begin{array}{c} \\ \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \\ \mathrm{d} \\ \mathrm{e} \\ \mathrm{f} \\ \mathrm{g} \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathrm{a} & \mathrm{b} & \mathrm{c} & \mathrm{d} & \mathrm{e} & \mathrm{f} & \mathrm{g} \end{array} \\ \begin{pmatrix} 3 & 2 & 1 & 4 & 5 & 6 & 7 \\ 1 & 3 & 2 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 5 & 4 & 7 & 2 & 3 & 6 \\ 1 & 2 & 5 & 4 & 3 & 6 & 7 \\ 1 & 3 & 6 & 5 & 2 & 4 & 7 \\ 1 & 2 & 7 & 4 & 5 & 6 & 3 \end{pmatrix} \end{array} \qquad \begin{array}{c} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \\ \mathrm{d} \\ \mathrm{e} \\ \mathrm{f} \\ \mathrm{g} \end{array} \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} = \overline{\pi^{\mathsf{T}};\overline{[\mathfrak{A}\cap P;\mathfrak{A}];\mathbb{T}}}$$

Fig. 8.4 A hardly interesting binary mapping and its invariant elements

In case $\mathfrak{A}$ is a group operation, the invariant elements together form the center of the group.

Next interesting are left- resp. right-neutral elements. The intention for a right-neutral element $n_r : X \longrightarrow \mathbb{1}$ is that application of $\mathfrak{A}$ to any pair (x, n_r) with x chosen arbitrarily results in x. In the relational setting with points x, n_r, this reads

$\mathfrak{A}^{\mathsf{T}};(x \otimes n_r) = \mathfrak{A}^{\mathsf{T}};(\pi;x \cap \rho;n_r) = x.$

When working in a group theory environment, n_r is usually called zero or unit element, depending on whether one works in an additive or multiplicative setting. A point-free formulation for all x simultaneously is

$\mathfrak{A}^{\mathsf{T}};(\mathbb{I} \otimes n_r;\mathbb{T}_{\mathbb{1}X}) = \mathfrak{A}^{\mathsf{T}};(\pi;\mathbb{I}_X \cap \rho;n_r;\mathbb{T}_{\mathbb{1}X}) = \mathbb{I}_X.$

This is a condition n_r has to satisfy. Concentrating on "$\subseteq$" alone, the following equivalences make it more explicit:

$\mathfrak{A}^{\mathsf{T}};(\pi;\mathbb{I} \cap \rho;n_r;\mathbb{T}) \subseteq \mathbb{I} \iff \pi \cap \rho;n_r;\mathbb{T} \subseteq \mathfrak{A} \iff \rho;n_r;\mathbb{T} \subseteq \mathfrak{A} \cup \overline{\pi}$

$\iff \rho^{\mathsf{T}};(\overline{\mathfrak{A}} \cap \pi) \subseteq \overline{n_r;\mathbb{T}} \iff n_r;\mathbb{T} \subseteq \overline{\rho^{\mathsf{T}};(\overline{\mathfrak{A}} \cap \pi)}$

The n_r thus characterized may in arbitrarily chosen cases uninterestingly be equal to $\mathbb{\perp}$ for which $\rho^{\mathsf{T}};\pi = \mathbb{T}$ gives a hint. We assume, however, a point $e \subseteq n_r$ and recall that according to Prop. 8.1 $g := (\pi \cap \rho;e;\mathbb{T})^{\mathsf{T}}$ is a map. From

$\mathfrak{A}^{\mathsf{T}};g^{\mathsf{T}} = \mathfrak{A}^{\mathsf{T}};(\pi \cap \rho;e;\mathbb{T}) \subseteq \mathfrak{A}^{\mathsf{T}};(\pi \cap \rho;n_r;\mathbb{T}) \subseteq \mathbb{I}$

we then derive equality: The mapping $g;\mathfrak{A}$ contained in the mapping $\mathbb{I}$ means that they are equal.

8.6 Definition. Let be given the binary mapping $\mathfrak{A}$ as before. We call any point e in

$\overline{\rho^{\mathsf{T}};(\overline{\mathfrak{A}} \cap \pi)}$ a **right-neutral** element,

$\overline{\pi^{\mathsf{T}};(\overline{\mathfrak{A}} \cap \rho)}$ a **left-neutral** element,

$\overline{\rho^{\mathsf{T}};(\overline{\mathfrak{A}} \cap \pi)} \cap \overline{\pi^{\mathsf{T}};(\overline{\mathfrak{A}} \cap \rho)}$ a **neutral** element. □

In an alternative approach, we might have considered

$$\delta_r := \mathbb{I}_{X\times X} \cap \mathfrak{A}{;}\pi^{\mathsf{T}} : X\times X \longrightarrow X \times X$$

i.e., all the pairs with result and left component equal. Then one would look for points e in

$$n_r := \overline{\rho^{\mathsf{T}}{;}\overline{\delta_r{;}\mathbb{T}_{X\times X,X}}} : X \longrightarrow X,$$

indicating right-neutral elements if any, and then giving rise to forming of right-inverses

$$i_r := \pi^{\mathsf{T}}{;}(\mathfrak{A}{;}e{;}\mathbb{T} \cap \rho) = \mathtt{rel}(\mathfrak{A}{;}e{;}\mathbb{T}) : X \longrightarrow X.$$

With the standard methods, it is possible to prove

$$\begin{array}{ll}
\multicolumn{2}{l}{\mathfrak{A}^{\mathsf{T}}{;}\,(i_r \ogreaterthan \mathbb{I}) = \mathfrak{A}^{\mathsf{T}}{;}(\pi{;}i_r \cap \rho) \subseteq e{;}\mathbb{T}} \\
\Longleftrightarrow & \mathfrak{A}{;}\overline{e{;}\mathbb{T}} \subseteq \overline{\pi{;}i_r \cap \rho} = \overline{\pi{;}i_r} \cup \overline{\rho} \\
\Longleftrightarrow & \mathbb{T} = \overline{\mathfrak{A}{;}\overline{e{;}\mathbb{T}}} \cup \overline{\pi{;}i_r} \cup \overline{\rho} = \mathfrak{A}{;}e{;}\mathbb{T} \cup \overline{\pi{;}i_r} \cup \overline{\rho} \quad \text{since } \mathfrak{A} \text{ is a map} \\
\Longleftrightarrow & \pi{;}i_r \subseteq \mathfrak{A}{;}e{;}\mathbb{T} \cup \overline{\rho} \\
\Longleftrightarrow & \pi^{\mathsf{T}}{;}\overline{\mathfrak{A}{;}e{;}\mathbb{T} \cup \overline{\rho}} \subseteq \overline{i_r} \\
\Longleftrightarrow & i_r \subseteq \overline{\pi^{\mathsf{T}}{;}(\overline{\mathfrak{A}{;}e{;}\mathbb{T}} \cap \rho)} = \overline{\mathtt{rel}(\overline{\mathfrak{A}{;}e{;}\mathbb{T}})} = \mathtt{rel}(\mathfrak{A}{;}e{;}\mathbb{T}) \quad \text{due to Prop. 6.4.vii}
\end{array}$$

We have to show equality $\mathfrak{A}^{\mathsf{T}}{;}\,(i_r \ogreaterthan \mathbb{I}) = \mathfrak{A}^{\mathsf{T}}{;}(\pi{;}i_r \cap \rho) = e{;}\mathbb{T}$ with a separate argument, based on the fact that e is a neutral point, or else, a transposed mapping. It suffices, according to Prop. 5.2.iii of [Sch11], when $\mathfrak{A}^{\mathsf{T}}{;}(\pi{;}i_r \cap \rho)$ turns out to be surjective

$$\begin{array}{l}
\mathbb{T}{;}\mathfrak{A}^{\mathsf{T}}{;}(\pi{;}i_r \cap \rho) = \mathbb{T}{;}(\pi{;}i_r \cap \rho) = \mathbb{T}{;}\rho^{\mathsf{T}}{;}(\pi{;}i_r \cap \rho) = \mathbb{T}{;}(\rho^{\mathsf{T}}{;}\pi{;}i_r \cap \mathbb{I}) = \mathbb{T}{;}(\mathbb{T}{;}i_r \cap \mathbb{I}) = \mathbb{T}, \quad \text{since} \\
\mathbb{T}{;}i_r = \mathbb{T}{;}\pi^{\mathsf{T}}{;}(\mathfrak{A}{;}e{;}\mathbb{T} \cap \rho) = \mathbb{T}{;}(\mathfrak{A}{;}e{;}\mathbb{T} \cap \rho) = (\mathbb{T} \cap \mathbb{T}{;}e^{\mathsf{T}}{;}\mathfrak{A}^{\mathsf{T}}){;}\rho = \mathbb{T}{;}e^{\mathsf{T}}{;}\mathfrak{A}^{\mathsf{T}}{;}\rho = \mathbb{T}{;}e^{\mathsf{T}}{;}\mathbb{T} = \mathbb{T}
\end{array}$$

when $\mathfrak{A}$ allows right-inversion and e is a point.

As an example, we show the alternating group A_3 as well as a constant binary mapping.

$$\begin{array}{c|c}
 & [1,2,3] \; [2,3,1] \; [3,1,2] \\
\hline
\begin{array}{c}[1,2,3]\\ [2,3,1]\\ [3,1,2]\end{array} & \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}
\end{array}
\qquad \mathfrak{A}$$

$$\begin{array}{l|ccccccccc}
 & ([1,2,3],[1,2,3]) & ([2,3,1],[1,2,3]) & ([1,2,3],[2,3,1]) & ([3,1,2],[1,2,3]) & ([2,3,1],[2,3,1]) & ([1,2,3],[3,1,2]) & ([3,1,2],[2,3,1]) & ([2,3,1],[3,1,2]) & ([3,1,2],[3,1,2]) \\
\hline
([1,2,3],[1,2,3]) & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
([2,3,1],[1,2,3]) & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
([1,2,3],[2,3,1]) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
([3,1,2],[1,2,3]) & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
([2,3,1],[2,3,1]) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
([1,2,3],[3,1,2]) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
([3,1,2],[2,3,1]) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
([2,3,1],[3,1,2]) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
([3,1,2],[3,1,2]) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\qquad \delta_r$$

$$\begin{array}{l|ccc}
 & [1,2,3] & [2,3,1] & [3,1,2] \\
\hline
[1,2,3] & 1 & 1 & 1 \\
[2,3,1] & 0 & 0 & 0 \\
[3,1,2] & 0 & 0 & 0
\end{array}
\quad n_r
\qquad\qquad
\begin{array}{l|ccc}
 & [1,2,3] & [2,3,1] & [3,1,2] \\
\hline
[1,2,3] & 1 & 0 & 0 \\
[2,3,1] & 0 & 0 & 1 \\
[3,1,2] & 0 & 1 & 0
\end{array}
\quad i_r$$

Fig. 8.5 Existence of right-neutral elements

Here also the forming of inverses i_r is indicated. Since n_r in Fig. 8.6 is not row-constant, it cannot contain a point, so that there is no right-neutral element.

$$
\begin{array}{c|cccc}
 & a & b & c & d \\
\hline
a & 1 & 1 & 1 & 1 \\
b & 1 & 1 & 1 & 1 \\
c & 1 & 1 & 1 & 1 \\
d & 1 & 1 & 1 & 1
\end{array}
\qquad
\begin{array}{c|cccccccccccccccc}
 & (a,a) & (b,a) & (a,b) & (c,a) & (b,b) & (a,c) & (d,a) & (c,b) & (b,c) & (a,d) & (d,b) & (c,c) & (b,d) & (d,c) & (c,d) & (d,d) \\
\hline
(a,a) & 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
(b,a) & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
(a,b) & 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
(c,a) & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
(b,b) & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
(a,c) & 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\
(d,a) & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
(c,b) & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
(b,c) & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
(a,d) & 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\
(d,b) & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
(c,c) & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
(b,d) & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
(d,c) & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
(c,d) & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
(d,d) & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}
\qquad
\begin{array}{c|cccc}
 & a & b & c & d \\
\hline
a & 1 & 0 & 0 & 0 \\
b & 1 & 0 & 0 & 0 \\
c & 1 & 0 & 0 & 0 \\
d & 1 & 0 & 0 & 0
\end{array}
\qquad
\begin{array}{c|cccc}
 & a & b & c & d \\
\hline
a & 1 & 0 & 0 & 0 \\
b & 1 & 0 & 0 & 0 \\
c & 1 & 0 & 0 & 0 \\
d & 1 & 0 & 0 & 0
\end{array}
$$

$\mathfrak{A}$ $\qquad$ δ_r $\qquad$ n_r $\qquad$ i_r

Fig. 8.6 Non-existence of right-neutral elements

Right- or left-neutral elements may exist or not. In Fig. 8.7 we see what it means to be right-neutral: The corresponding two columns correspond to the row-inscriptions.

$$
\mathfrak{A} = \begin{array}{c|ccccccc}
 & a & b & c & d & e & f & g \\
\hline
a & 3 & 2 & 1 & 4 & 1 & 6 & 7 \\
b & 1 & 3 & 2 & 4 & 2 & 6 & 7 \\
c & 1 & 2 & 3 & 4 & 3 & 6 & 7 \\
d & 1 & 5 & 4 & 7 & 4 & 3 & 6 \\
e & 1 & 2 & 5 & 4 & 5 & 6 & 7 \\
f & 1 & 3 & 6 & 5 & 6 & 4 & 7 \\
g & 1 & 2 & 7 & 4 & 7 & 6 & 3
\end{array}
\qquad
\begin{array}{c|c}
a & 0 \\
b & 0 \\
c & 1 \\
d & 0 \\
e & 1 \\
f & 0 \\
g & 0
\end{array} = \mathfrak{n}_{\text{right}}
\qquad
\begin{array}{c|ccccccc}
 & a & b & c & d & e & f & g \\
\hline
a & 1&0&0&0&0&0&0 \\
b & 0&1&0&0&0&0&0 \\
c & 0&0&1&0&1&0&0 \\
d & 0&1&0&0&0&1&0 \\
e & 0&0&1&0&1&0&0 \\
f & 0&1&0&1&0&0&0 \\
g & 0&0&0&0&0&0&1
\end{array}
\qquad
\begin{array}{c|ccccccc}
 & a & b & c & d & e & f & g \\
\hline
a & 0&0&0&0&0&0&0 \\
b & 0&0&0&0&0&0&0 \\
c & 0&0&0&0&0&0&0 \\
d & 0&1&0&0&0&0&0 \\
e & 0&0&1&0&1&0&0 \\
f & 0&0&0&1&0&0&0 \\
g & 0&0&0&0&0&0&0
\end{array}
$$

Fig. 8.7 Binary map without left- but two right-neutrals c, e and right-inverses wrt. to c and e

A left-neutral element in analogy, gives rise to a row identical with the column numbering. From this fact it will become clear that there can be at most one point as neutral element e. The aforementioned transition to inverses

$$i := \pi^{\mathsf{T}}(\mathfrak{A}; e \cap \rho).$$

will then be a bijective mapping, which it was neither for c nor for e in Fig. 8.7.

8.7 Proposition. For some binary mapping $\mathfrak{A}$ we consider the left- as well as right-neutral element sets n_l, n_r. If both contain points e_l, e_r, these will be equal.

Proof: We apply the result obtained before in two directions

$$\mathfrak{A}^{\mathsf{T}}; (\pi; Y \cap \rho; e_r; \mathbb{T}) = \mathfrak{A}^{\mathsf{T}}; (\pi \cap \rho; e_r; \mathbb{T}); Y = \mathfrak{A}^{\mathsf{T}}; g^{\mathsf{T}}; Y = \mathbb{I}; Y = Y,$$

and correspondingly

$$\mathfrak{A}^{\mathsf{T}}; (\pi; e_l; \mathbb{T} \cap \rho; Z) = \mathfrak{A}^{\mathsf{T}}; (\rho \cap \pi; e_l; \mathbb{T}); Z = \mathfrak{A}^{\mathsf{T}}; f^{\mathsf{T}}; Z = \mathbb{I}; Z = Z.$$

Therefore

$$e_l;\mathbb{T} = \mathfrak{A}^\mathsf{T}; (e_l;\mathbb{T} \circledgtr e_r;\mathbb{T}) = \mathfrak{A}^\mathsf{T}; (\pi;e_l;\mathbb{T} \cap \rho;e_r;\mathbb{T}) = e_r;\mathbb{T} \qquad \square$$

Should there exist more than one in either one of n_l, n_r they will thus all be equal.

$$\begin{tikzcd}
(X\times X)\times X \arrow[d, "\pi'"'] \arrow[dd, bend left, "\rho'"] \\
X\times X \arrow[d, bend right, "\pi"'] \arrow[d, "\mathcal{J}"'] \arrow[d, "\mathcal{M}"] \arrow[d, bend left, "\rho"] \\
X
\end{tikzcd}$$

Fig. 8.8 Illustrating distributivity

Also the concept of distributivity may be formulated relationally in case there are two binary mappings $\mathfrak{J}$, $\mathfrak{M}$, as, e.g., in a lattice the join and meet.

8.8 Definition. Given two binary mappings, we say that $\mathfrak{J}$ **distributes** over $\mathfrak{M}$, when

$$((\pi \otimes \mathbb{I});\mathfrak{J} \circledless (\rho \otimes \mathbb{I});\mathfrak{J});\mathfrak{M} = (\mathfrak{M} \otimes \mathbb{I});\mathfrak{J},$$

or else, when $\hat{\mathfrak{J}};\mathfrak{M} = (\mathfrak{M} \otimes \mathbb{I});\mathfrak{J}$ as we will later slightly abbreviate. $\square$

One might also demand in blown-up form resembling $(a \vee c) \wedge (b \vee c) = (a \wedge b) \vee c$

$$\big[(\pi';\pi;\pi^\mathsf{T} \cap \rho';\rho^\mathsf{T});\mathfrak{J};\pi^\mathsf{T} \cap (\pi';\rho;\pi^\mathsf{T} \cap \rho';\rho^\mathsf{T});\mathfrak{J};\rho^\mathsf{T}\big];\mathfrak{M} = (\pi';\mathfrak{M};\pi^\mathsf{T} \cap \rho';\rho^\mathsf{T});\mathfrak{J}.$$

9 Boolean algebras

A note seems necessary concerning Boolean algebras; here supported with visualization in a concrete example. The peculiar recursive and fractal symmetries of these examples often give additional insight — and have already triggered secretaries to stitch such patterns for a pot cloth.

Most people work with subsets $U \subseteq X$, while we distinguish between a subset in this standard form and the corresponding element e in the powerset, considered as a point. The two are related via the membership relation ε as shown in Fig. 9.1 together with the powerset ordering $\Omega = \overline{\varepsilon^\mathsf{T};\overline{\varepsilon}}$.

Theoreticians frequently consider Boolean algebras "with signature $\langle X, \cdot, +, -, \mathbf{0}, \mathbf{1} \rangle$". Following their idea, we find on X the operations $\cap, \cup, \overline{}, \mathbb{⊥}, \mathbb{T}$.

$$U = \varepsilon;e \qquad e = \mathtt{syq}(\varepsilon, U)$$

$$\begin{array}{c|cccccccccccccccc|}
 & \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \\
\mathrm{a} & 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\
\mathrm{b} & 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\
\mathrm{c} & 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\
\mathrm{d} & 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1
\end{array} \quad \begin{pmatrix} 0\\1\\0\\1 \end{pmatrix}$$

$$(0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0) = e^{\mathsf{T}}$$

$$\begin{array}{c|cccccccccccccccc|}
 & \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \\
\{\} & 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
\{a\} & 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\
\{b\} & 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\
\{a,b\} & 0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1 \\
\{c\} & 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\
\{a,c\} & 0&0&0&0&0&1&0&1&0&0&0&0&0&1&0&1 \\
\{b,c\} & 0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1 \\
\{a,b,c\} & 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1 \\
\{d\} & 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\
\{a,d\} & 0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1 \\
\{b,d\} & 0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1 \\
\{a,b,d\} & 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1 \\
\{c,d\} & 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \\
\{a,c,d\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1 \\
\{b,c,d\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1 \\
\{a,b,c,d\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{array} \quad \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0 \end{pmatrix}$$

Fig. 9.1 Subset U and corresponding point e in the powerset via ε, Ω

There is, however, a second "lifted" form, for which the elements are taken from $\mathbf{2}^X$ with corresponding operations consisting of

$$\mathfrak{M}, \quad \mathfrak{J}, \quad N, \quad (\overline{\varepsilon^{\mathsf{T}};\mathbb{T}} =)\, \mathtt{syq}(\varepsilon, \mathbb{\perp}), \quad (\overline{\overline{\varepsilon}^{\mathsf{T}};\mathbb{T}} =)\, \mathtt{syq}(\varepsilon, \mathbb{T}),$$

as defined below. Easiest to observe are the 0-ary operators or elements $\overline{\varepsilon^{\mathsf{T}};\mathbb{T}} \approx \mathbf{0}, \overline{\overline{\varepsilon}^{\mathsf{T}};\mathbb{T}} \approx \mathbf{1}$ for which obviously, looking at Fig. 9.1,

$$\mathbb{\perp} = \varepsilon;\overline{\varepsilon^{\mathsf{T}};\mathbb{T}} = \varepsilon;\mathtt{syq}(\varepsilon, \mathbb{\perp}), \qquad \mathbb{T} = \varepsilon;\overline{\overline{\varepsilon}^{\mathsf{T}};\mathbb{T}} = \varepsilon;\mathtt{syq}(\varepsilon, \mathbb{T}).$$

Next we study the unary operator

$$N := \mathtt{syq}(\overline{\varepsilon}, \varepsilon) \qquad\qquad N : \mathbf{2}^X \longrightarrow \mathbf{2}^X,$$

visualized in Fig. 9.2, for which we show in advance

$$\overline{\varepsilon};N = \overline{\varepsilon};\mathtt{syq}(\overline{\varepsilon}, \varepsilon) = \varepsilon \qquad\qquad \varepsilon;N = \varepsilon;\mathtt{syq}(\overline{\varepsilon}, \varepsilon) = \varepsilon;\mathtt{syq}(\varepsilon, \overline{\varepsilon}) = \overline{\varepsilon}$$

$$\mathbb{I} \subseteq \Omega = \overline{\varepsilon^{\mathsf{T}};\overline{\varepsilon}} = \overline{\varepsilon^{\mathsf{T}};\varepsilon;N} \quad \Longrightarrow \quad N \subseteq \overline{\varepsilon^{\mathsf{T}};\varepsilon}.$$

Multiplying a relation with N from the left flips this relation upside/down, while multiplying from the right side flips it left/right. Sometimes, we have to apply N to both sides of a pair, for which purpose we also introduce

$$\mathcal{N} := (N \otimes N) = \pi;N;\pi^{\mathsf{T}} \cap \rho;N;\rho^{\mathsf{T}} : \mathbf{2}^X \times \mathbf{2}^X \longrightarrow \mathbf{2}^X \times \mathbf{2}^X.$$

We identify here disjointness $\overline{\varepsilon^{\mathsf{T}};\varepsilon}$ which is shown in Fig. 9.2. It looks as if the powerset ordering Ω of Fig. 9.1 were rotated by an angle of -90 degrees, which may more mathematically be expressed as $\Omega;N = \overline{\varepsilon^{\mathsf{T}};\varepsilon}$; this time flipping left/right.

$$
\begin{array}{c|cccccccccccccccc}
 & \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \\
\hline
\{\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\
\{a\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\
\{b\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\
\{a,b\} & 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\
\{c\} & 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\
\{a,c\} & 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\
\{b,c\} & 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\
\{a,b,c\} & 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\
\{d\} & 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\
\{a,d\} & 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\
\{b,d\} & 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\
\{a,b,d\} & 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\
\{c,d\} & 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{a,c,d\} & 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{b,c,d\} & 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{a,b,c,d\} & 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}
\qquad
\begin{array}{c|cccccccccccccccc}
 & \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} & \{d\} & \{a,d\} & \{b,d\} & \{a,b,d\} & \{c,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \\
\hline
\{\} & 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
\{a\} & 1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0 \\
\{b\} & 1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0 \\
\{a,b\} & 1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0 \\
\{c\} & 1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0 \\
\{a,c\} & 1&0&1&0&0&0&0&0&1&0&1&0&0&0&0&0 \\
\{b,c\} & 1&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0 \\
\{a,b,c\} & 1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\
\{d\} & 1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0 \\
\{a,d\} & 1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0 \\
\{b,d\} & 1&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0 \\
\{a,b,d\} & 1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\
\{c,d\} & 1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{a,c,d\} & 1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{b,c,d\} & 1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{a,b,c,d\} & 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}
$$

Fig. 9.2 Negation N and disjointness $\overline{\varepsilon^{\mathsf{T}};\varepsilon} = \Omega;N$ in the powerset

At last, we consider the binary operations meet $\mathfrak{M}$ and join $\mathfrak{J}$ which we mainly obtain specializing the result of Prop. 7.8 to the case $X = Y$ and integrate them into the relational mechanism using the least upper, resp. greatest lower, bound taken rowwise according to [Sch11] Prop. 9.10.

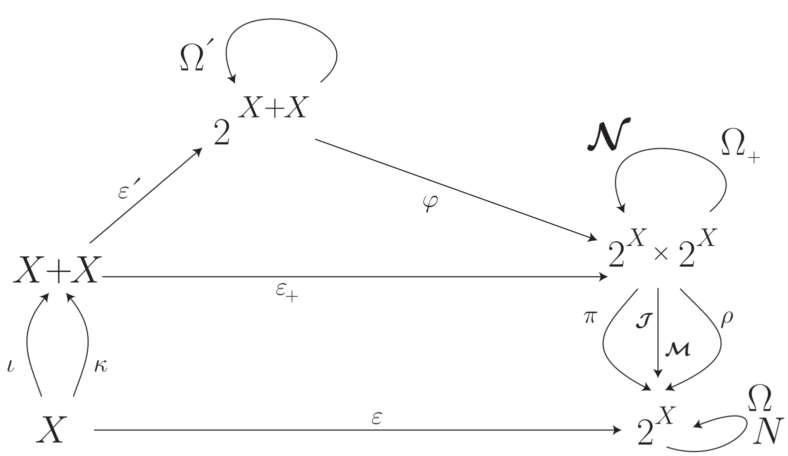

Fig. 9.3 Converting subsets of a sum to products of subsets with join $\mathfrak{J}$ and meet $\mathfrak{M}$

A first step is the investigation of the bijection φ of Figs. 9.3 and 9.4. We show the relation indicating with >a, respectively a< whether an element has been injected to the left or to the right. Only when restricting to somehow coherent visualizations of $\mathbf{2}^{X+X}$ and $\mathbf{2}^X \times \mathbf{2}^X$, this will show a 'diagonal'.

9.1 Proposition. We assume the setting of Prop. 7.8, however with $X = Y$, so that additional formulae may be formulated including join and meet.

i) $\mathfrak{J} = \mathtt{syq}(\iota;\varepsilon_+ \cup \kappa;\varepsilon_+, \varepsilon) = \mathtt{syq}(\varepsilon;\pi^{\mathsf{T}} \cup \varepsilon;\rho^{\mathsf{T}}, \varepsilon) = \mathtt{lubR}_\Omega(\pi \cup \rho) = \mathtt{syq}(\varepsilon;[\pi \cup \rho]^{\mathsf{T}}, \varepsilon)$

$= \mathtt{syq}(\,(\overline{\varepsilon} \otimes \overline{\varepsilon})\,, \overline{\varepsilon}) = \mathtt{syq}(\overline{\varepsilon};\pi^{\mathsf{T}} \cap \overline{\varepsilon};\rho^{\mathsf{T}}, \overline{\varepsilon})$

ii) $\mathfrak{M} = \mathtt{syq}(\iota;\varepsilon_+ \cap \kappa;\varepsilon_+, \varepsilon) = \mathtt{syq}(\varepsilon;\pi^\mathsf{T} \cap \varepsilon;\rho^\mathsf{T}, \varepsilon) = \mathtt{glbR}_\Omega(\pi \cup \rho) = \mathtt{syq}(\overline{\varepsilon};[\pi \cup \rho]^\mathsf{T}, \overline{\varepsilon})$

$= \mathtt{syq}(\,(\varepsilon \otimes \varepsilon)\,, \varepsilon)$

iii) $\varepsilon;\mathfrak{J}^\mathsf{T} = \iota;\varepsilon_+ \cup \kappa;\varepsilon_+ = \varepsilon;\pi^\mathsf{T} \cup \varepsilon;\rho^\mathsf{T}$

iv) $\varepsilon;\mathfrak{M}^\mathsf{T} = \iota;\varepsilon_+ \cap \kappa;\varepsilon_+ = \varepsilon;\pi^\mathsf{T} \cap \varepsilon;\rho^\mathsf{T} = (\varepsilon \otimes \varepsilon)$

$$
\begin{array}{r|cccccccccccccccc|}
 & (\{\},\{\}) & (\{a\},\{\}) & (\{\},\{a\}) & (\{b\},\{\}) & (\{a\},\{a\}) & (\{\},\{b\}) & (\{a,b\},\{\}) & (\{b\},\{a\}) & (\{a\},\{b\}) & (\{\},\{a,b\}) & (\{a,b\},\{a\}) & (\{b\},\{b\}) & (\{a\},\{a,b\}) & (\{a,b\},\{b\}) & (\{b\},\{a,b\}) & (\{a,b\},\{a,b\}) \\
\{\} & 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{a<\} & 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{>a\} & 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{a<,>a\} & 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\
\{b<\} & 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\
\{a<,b<\} & 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\
\{>a,b<\} & 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\
\{a<,>a,b<\} & 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\
\{>b\} & 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\
\{a<,>b\} & 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\
\{>a,>b\} & 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\
\{a<,>a,>b\} & 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\
\{b<,>b\} & 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\
\{a<,b<,>b\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\
\{>a,b<,>b\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\
\{a<,>a,b<,>b\} & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{array}
$$

Fig. 9.4 Relation φ converting subsets of a sum to products of subsets for $X := \{a, b\}$

Proof: i) We formulate the join $\mathfrak{J}$ as a least upper bound and recall Prop. 9.10 of [Sch11]
$\mathfrak{J}^\mathsf{T} = \mathtt{lub}_\Omega([\pi \cup \rho]^\mathsf{T}) = \mathtt{syq}(\varepsilon, \varepsilon;[\pi \cup \rho]^\mathsf{T}) = \mathtt{syq}(\varepsilon, \varepsilon;\pi^\mathsf{T} \cup \varepsilon;\rho^\mathsf{T}) = \mathtt{syq}(\varepsilon, \iota;\varepsilon_+ \cup \kappa;\varepsilon_+)$

ii) $\mathfrak{M}^\mathsf{T} = \mathtt{glb}_\Omega([\pi \cup \rho]^\mathsf{T}) = \mathtt{syq}(\overline{\varepsilon}, \overline{\varepsilon};[\pi \cup \rho]^\mathsf{T}) = \mathtt{syq}(\varepsilon, \varepsilon;\pi^\mathsf{T} \cap \varepsilon;\rho^\mathsf{T}) = \mathtt{syq}(\varepsilon, \iota;\varepsilon_+ \cap \kappa;\varepsilon_+)$

$\mathfrak{M} = \mathtt{glbR}_\Omega(\pi \cup \rho)$ by definition
$= \left[\mathtt{glb}_\Omega([\pi \cup \rho]^\mathsf{T})\right]^\mathsf{T}$ by definition
$= \left[\mathtt{syq}(\overline{\varepsilon}, \overline{\varepsilon};[\pi \cup \rho]^\mathsf{T})\right]^\mathsf{T}$ Prop. 9.10 of [Sch11]
$= \mathtt{syq}(\overline{\varepsilon};\pi^\mathsf{T} \cup \overline{\varepsilon};\rho^\mathsf{T}, \overline{\varepsilon})$
$= \mathtt{syq}(\overline{\overline{\varepsilon};\pi^\mathsf{T} \cup \overline{\varepsilon};\rho^\mathsf{T}}, \varepsilon)$ Prop. 8.10.i of [Sch11]
$= \mathtt{syq}(\overline{\overline{\varepsilon};\pi^\mathsf{T}} \cap \overline{\overline{\varepsilon};\rho^\mathsf{T}}, \varepsilon)$
$= \mathtt{syq}(\overline{\overline{\varepsilon}};\pi^\mathsf{T} \cap \overline{\overline{\varepsilon}};\rho^\mathsf{T}, \varepsilon)$
$= \mathtt{syq}(\varepsilon;\pi^\mathsf{T} \cap \varepsilon;\rho^\mathsf{T}, \varepsilon)$

$\mathfrak{J} = \mathtt{lubR}_\Omega(\pi \cup \rho)$ by definition
$= \left[\mathtt{lub}_\Omega([\pi \cup \rho]^\mathsf{T})\right]^\mathsf{T}$ by definition
$= \left[\mathtt{syq}(\varepsilon, \varepsilon;[\pi \cup \rho]^\mathsf{T})\right]^\mathsf{T}$ Prop. 9.10 of [Sch11]
$= \mathtt{syq}(\varepsilon;\pi^\mathsf{T} \cup \varepsilon;\rho^\mathsf{T}, \varepsilon)$

iv) $\mathfrak{M};\varepsilon^\mathsf{T} = \left[\varepsilon;\mathfrak{M}^\mathsf{T}\right]^\mathsf{T} = \left[\varepsilon;\mathtt{syq}(\varepsilon, \varepsilon;\pi^\mathsf{T} \cap \varepsilon;\rho^\mathsf{T})\right]^\mathsf{T} = \left[\varepsilon;\pi^\mathsf{T} \cap \varepsilon;\rho^\mathsf{T}\right]^\mathsf{T} = \pi;\varepsilon^\mathsf{T} \cap \rho;\varepsilon^\mathsf{T} = (\varepsilon \oslash \varepsilon)$

iii) The proof for $\mathfrak{J};\varepsilon^\mathsf{T}$ is established in a similar way. $\square$

We convince us formally that $\mathfrak{M}$ is commutative:

$$
\begin{aligned}
P ; \mathfrak{M} &= P ; \mathtt{syq}(\,(\varepsilon \olessthan \varepsilon)\,,\varepsilon) \quad \text{by definition}\\
&= \mathtt{syq}(\,(\varepsilon \olessthan \varepsilon) ; P, \varepsilon) \quad \text{since } P \text{ is a bijective mapping}\\
&= \mathtt{syq}(\varepsilon ; \pi^\top ; P \cap \varepsilon ; \rho^\top ; P, \varepsilon)\\
&= \mathtt{syq}(\varepsilon ; \rho^\top \cap \varepsilon ; \pi^\top, \varepsilon)\\
&= \mathtt{syq}(\,(\varepsilon \olessthan \varepsilon)\,,\varepsilon) = \mathfrak{M}
\end{aligned}
$$

A trivial remark is in order, namely that a pair with coinciding first and second component will have precisely this coinciding set as its meet, i.e.

$$\pi \cap \rho \subseteq \mathfrak{M} \quad \text{or} \quad (\mathbb{I} \ogreaterthan \mathbb{I}) \subseteq \mathfrak{M}\,.$$

The proof can also be carried out in a fully formal way:

$$
\begin{aligned}
&\iff \quad \pi \cap \rho \subseteq \mathtt{syq}(\varepsilon ; \pi^\top \cap \varepsilon ; \rho^\top, \varepsilon)\\
&\iff \quad \overline{\varepsilon ; \pi^\top \cap \varepsilon ; \rho^\top}^{\top} ; \varepsilon \cup (\varepsilon ; \pi^\top \cap \varepsilon ; \rho^\top)^\top ; \overline{\varepsilon} \subseteq \overline{\pi} \cup \overline{\rho}\\
&\iff \quad (\pi \cap \rho) ; \varepsilon^\top \subseteq (\varepsilon ; \pi^\top \cap \varepsilon ; \rho^\top)^\top \quad \text{and} \quad (\pi \cap \rho) ; \overline{\varepsilon}^\top \subseteq \overline{\varepsilon ; \pi^\top \cap \varepsilon ; \rho^\top}^{\top}\\
&\iff \quad (\pi \cap \rho) ; \varepsilon^\top \subseteq \pi ; \varepsilon^\top \cap \rho ; \varepsilon^\top \quad \text{and} \quad (\pi \cap \rho) ; \overline{\varepsilon}^\top \subseteq \overline{\pi ; \varepsilon^\top} \cup \overline{\rho ; \varepsilon^\top} = \pi ; \overline{\varepsilon}^\top \cup \rho ; \overline{\varepsilon}^\top \text{ which is true.}
\end{aligned}
$$

Some other helpful formulae:

9.2 Proposition.

i) $\mathcal{N} ; \pi = \pi ; N, \quad \mathcal{N} ; \rho = \rho ; N, \qquad \mathcal{N} ; \mathfrak{M} = \mathfrak{J} ; N, \quad \mathcal{N} ; \mathfrak{J} = \mathfrak{M} ; N$

ii) $\mathfrak{M}^\top ; \pi = \Omega \qquad \mathfrak{M}^\top ; \rho = \Omega$

iii) $\mathfrak{J}^\top ; \pi = \Omega^\top \qquad \mathfrak{J}^\top ; \rho = \Omega^\top$

iv) $\mathfrak{M} ; \Omega^\top = \pi ; \Omega^\top \cap \rho ; \Omega^\top = (\Omega^\top \ogreaterthan \Omega^\top) \qquad \mathfrak{J} ; \Omega = \pi ; \Omega \cap \rho ; \Omega = (\Omega \ogreaterthan \Omega)$

v) $(\varepsilon ; \pi^\top \cap \varepsilon ; \rho^\top) ; \mathfrak{M} = \varepsilon \qquad (\varepsilon ; \pi^\top \cup \varepsilon ; \rho^\top) ; \mathfrak{J} = \varepsilon$ variant form $(\varepsilon \olessthan \varepsilon) ; \mathfrak{M} = \varepsilon$

vi) $(\varepsilon \olessthan \varepsilon) ; (\Omega \otimes \Omega) = (\varepsilon \olessthan \varepsilon)$

vii) $\mathfrak{M}^\top ; \mathtt{syq}((\varepsilon ; \pi^\top \cap \varepsilon ; \rho^\top), X) = \mathtt{syq}((\varepsilon ; \pi^\top \cap \varepsilon ; \rho^\top) ; \mathfrak{M}, X)$

$\mathfrak{M}^\top ; \mathtt{syq}(\,(\varepsilon \olessthan \varepsilon)\,, X) = \mathtt{syq}(\,(\varepsilon \olessthan \varepsilon) ; \mathfrak{M}, X)$

viii) $\pi ; \Omega \cap \rho \subseteq \mathfrak{J} \qquad \rho ; \Omega \cap \pi \subseteq \mathfrak{J}$ or in variant form

$(\Omega \ogreaterthan \mathbb{I}) \subseteq \mathfrak{J} \qquad (\mathbb{I} \ogreaterthan \Omega) \subseteq \mathfrak{J}$

Proof: i) Since $N, \mathcal{N}$ are mappings, we may apply Prop. 7.2.ii to the first two and then proceed with, e.g.

$$
\begin{aligned}
&\mathcal{N} ; \mathfrak{M} ; N = \mathcal{N} ; \mathtt{syq}(\varepsilon ; \pi^\top \cap \varepsilon ; \rho^\top, \varepsilon) ; N = \mathtt{syq}([\varepsilon ; \pi^\top \cap \varepsilon ; \rho^\top] ; \mathcal{N}^\top, \varepsilon ; N)\\
&= \mathtt{syq}(\varepsilon ; \pi^\top ; \mathcal{N}^\top \cap \varepsilon ; \rho^\top ; \mathcal{N}^\top, \overline{\varepsilon}) = \mathtt{syq}(\varepsilon ; \pi^\top ; \mathcal{N} \cap \varepsilon ; \rho^\top ; \mathcal{N}, \overline{\varepsilon})\\
&= \mathtt{syq}(\varepsilon ; N ; \pi^\top \cap \varepsilon ; N ; \rho^\top, \overline{\varepsilon}) = \mathtt{syq}(\overline{\varepsilon} ; \pi^\top \cap \overline{\varepsilon} ; \rho^\top, \overline{\varepsilon})\\
&= \mathtt{syq}(\overline{\varepsilon ; \pi^\top} \cap \overline{\varepsilon ; \rho^\top}, \overline{\varepsilon}) = \mathtt{syq}(\overline{\varepsilon ; \pi^\top \cup \varepsilon ; \rho^\top}, \overline{\varepsilon}) = \mathtt{syq}(\varepsilon ; \pi^\top \cup \varepsilon ; \rho^\top, \varepsilon) = \mathfrak{J}
\end{aligned}
$$

ii) $\mathfrak{M}^\top = \mathtt{syq}(\varepsilon, \varepsilon ; \pi^\top \cap \varepsilon ; \rho^\top) = \overline{\overline{\varepsilon}^\top ; (\varepsilon ; \pi^\top \cap \varepsilon ; \rho^\top)} \cap \overline{\varepsilon^\top ; \overline{\varepsilon ; \pi^\top \cap \varepsilon ; \rho^\top}}$

$= \overline{\overline{\varepsilon}^\top;(\varepsilon;\pi^\top \cap \varepsilon;\rho^\top)} \cap \overline{\varepsilon^\top;\overline{\varepsilon;\pi^\top}} \cap \overline{\varepsilon^\top;\overline{\varepsilon;\rho^\top}} = \overline{\overline{\varepsilon}^\top;(\varepsilon;\pi^\top \cap \varepsilon;\rho^\top)} \cap \Omega;\pi^\top \cap \Omega;\rho^\top$

Now

$\mathfrak{M}^\top;\pi = \left[\overline{\overline{\varepsilon}^\top;(\varepsilon;\pi^\top \cap \varepsilon;\rho^\top)} \cap \Omega;\pi^\top \cap \Omega;\rho^\top\right];\pi$

$= \left[\Omega;\pi^\top \cap \left\{\overline{\overline{\varepsilon}^\top;(\varepsilon;\pi^\top \cap \varepsilon;\rho^\top)} \cap \Omega;\rho^\top\right\}\right];\pi = \Omega \cap \left\{\overline{\overline{\varepsilon}^\top;(\varepsilon;\pi^\top \cap \varepsilon;\rho^\top)} \cap \Omega;\rho^\top\right\};\pi = \Omega \cap \mathbb{T} = \Omega$

since

$\left\{\overline{\overline{\varepsilon}^\top;(\varepsilon;\pi^\top \cap \varepsilon;\rho^\top)} \cap \Omega;\rho^\top\right\};\pi \supseteq \left\{\overline{\overline{\varepsilon}^\top;\varepsilon;\rho^\top} \cap \Omega;\rho^\top\right\};\pi = \left\{\overline{\overline{\varepsilon}^\top;\varepsilon};\rho^\top \cap \Omega;\rho^\top\right\};\pi = \left\{\overline{\overline{\varepsilon}^\top;\varepsilon} \cap \Omega\right\};\rho^\top;\pi$

$= \left\{\Omega^\top \cap \Omega\right\};\rho^\top;\pi = \mathbb{I};\mathbb{T} = \mathbb{T}$

iii) $\mathfrak{J}^\top;\pi = N;\mathfrak{M}^\top;\mathcal{N};\pi = N;\mathfrak{M}^\top;\pi;N = N;\Omega;N = N;\overline{\varepsilon^\top;\overline{\varepsilon}};N = \overline{N;\varepsilon^\top;\overline{\varepsilon;N}} = \overline{\overline{\varepsilon}^\top;\overline{\overline{\varepsilon}}} = \overline{\overline{\varepsilon}^\top;\varepsilon} = \Omega^\top$

iv) From Prop. 9.1.iv, we have $\mathfrak{M};\varepsilon^\top = \pi;\varepsilon^\top \cap \rho;\varepsilon^\top$. Negation and multiplication with ε from the right side gives

$\overline{\mathfrak{M};\varepsilon^\top};\varepsilon = \overline{\pi;\varepsilon^\top};\varepsilon \cup \overline{\rho;\varepsilon^\top};\varepsilon$

$\iff \quad \overline{\overline{\mathfrak{M};\varepsilon^\top};\varepsilon} = \overline{\overline{\pi;\varepsilon^\top};\varepsilon} \cap \overline{\overline{\rho;\varepsilon^\top};\varepsilon}$

$\iff \quad \mathfrak{M};\overline{\overline{\varepsilon}^\top;\varepsilon} = \pi;\overline{\overline{\varepsilon}^\top;\varepsilon} \cap \rho;\overline{\overline{\varepsilon}^\top;\varepsilon}$

$\iff \quad \mathfrak{M};\Omega^\top = \pi;\Omega^\top \cap \rho;\Omega^\top$ meaning the intersection of lower cones

Alternative proof:

$\Omega;\mathfrak{M}^\top = \Omega;\mathtt{syq}(\varepsilon, (\varepsilon \varolessthan \varepsilon)) = \overline{\varepsilon^\top;\overline{\varepsilon}};\mathtt{syq}(\varepsilon, (\varepsilon \varolessthan \varepsilon))$

$= \overline{\varepsilon^\top;\overline{\varepsilon;\mathtt{syq}(\varepsilon, (\varepsilon \varolessthan \varepsilon))}}$ since every $\mathtt{syq}(\varepsilon, \ldots)$ is a transposed mapping

$= \overline{\varepsilon^\top;\overline{(\varepsilon \varolessthan \varepsilon)}} = \varepsilon\backslash (\varepsilon \varolessthan \varepsilon) = (\varepsilon\backslash\varepsilon \varolessthan \varepsilon\backslash\varepsilon)$ due to Prop. 7.6

$= (\Omega \varolessthan \Omega)$

v) $(\varepsilon;\pi^\top \cap \varepsilon;\rho^\top);\mathfrak{M} = (\varepsilon;\pi^\top \cap \varepsilon;\rho^\top);\mathtt{syq}(\varepsilon;\pi^\top \cap \varepsilon;\rho^\top, \varepsilon) = \varepsilon$,

since $\mathfrak{M}$ is surjective according to Prop. 9.4.ii and [Sch11] 8.12.iii; for $\mathfrak{J}$ similarly.

vi) The following is shown in two steps:

$(\varepsilon \varolessthan \varepsilon);(\Omega \otimes \Omega) = (\varepsilon;\pi^\top \cap \varepsilon;\rho^\top);(\pi;\Omega;\pi^\top \cap \rho;\Omega;\rho^\top)$

$\subseteq \varepsilon;\pi^\top;\pi;\Omega;\pi^\top \cap \varepsilon;\rho^\top;\rho;\Omega;\rho^\top$ isotony

$\subseteq \varepsilon;\Omega;\pi^\top \cap \varepsilon;\Omega;\rho^\top$ π, ρ are univalent

$= \varepsilon;\pi^\top \cap \varepsilon;\rho^\top = (\varepsilon \varolessthan \varepsilon)$ since $\varepsilon;\Omega = \varepsilon$

Short alternative proof:

$(\varepsilon \varolessthan \varepsilon);(\Omega \otimes \Omega) \subseteq (\varepsilon;\Omega \varolessthan \varepsilon;\Omega) = (\Omega \varolessthan \Omega)$ using Prop. 7.3.i

On the other hand side

$(\Omega \otimes \Omega) = \pi;\Omega;\pi^\top \cap \rho;\Omega;\rho^\top \supseteq \pi;\pi^\top \cap \rho;\rho^\top = \mathbb{I}$,

so that also

$(\varepsilon \varolessthan \varepsilon);(\Omega \otimes \Omega) \supseteq (\varepsilon \varolessthan \varepsilon);\mathbb{I} = (\varepsilon \varolessthan \varepsilon)$.

vii) We apply Prop. 8.18 of [Sch11] and, therefore, prove just

$(\varepsilon;\pi^\top \cap \varepsilon;\rho^\top);\mathfrak{M};\mathfrak{M}^\top = (\varepsilon;\pi^\top \cap \varepsilon;\rho^\top);\mathtt{syq}(\varepsilon;\pi^\top \cap \varepsilon;\rho^\top, \varepsilon);\mathfrak{M}^\top = \varepsilon;\mathfrak{M}^\top = (\varepsilon;\pi^\top \cap \varepsilon;\rho^\top)$

viii) $\pi;\Omega \cap \rho;\mathbb{I} = \pi;\Omega \cap \rho;(\Omega \cap \Omega^\top) = \pi;\Omega \cap \rho;\Omega \cap \rho;\Omega^\top$

$= \mathfrak{J};\Omega \cap \rho;\Omega^\top$ due to (iv)

$$\subseteq \mathfrak{J}; \Omega \cap \mathfrak{J}; \Omega^\mathsf{T} = \mathfrak{J}; (\Omega \cap \Omega^\mathsf{T}) = \mathfrak{J} \quad \text{since } \rho; \Omega^\mathsf{T} \subseteq \mathfrak{J}; \Omega^\mathsf{T} \iff \mathfrak{J}^\mathsf{T}; \rho; \Omega^\mathsf{T} = \Omega^\mathsf{T}; \Omega^\mathsf{T} \subseteq \Omega^\mathsf{T} \qquad \square$$

Of course, the traditional reasoning with orderings, e.g., $a \leq c, a \leq d \implies a \leq c \cap d$, assumes another shape.

9.3 Proposition. i) For points a, c, d we have

$$\begin{array}{l} a \subseteq \Omega; c \\ a \subseteq \Omega; d \end{array} \implies a \subseteq \Omega; \mathfrak{M}^\mathsf{T}; (c ⧁ d) \;=\; (\Omega ⧀ \Omega); (c ⧁ d)$$

ii) For points b, c, d we have

$$\begin{array}{l} b \subseteq \Omega^\mathsf{T}; c \\ b \subseteq \Omega^\mathsf{T}; d \end{array} \implies b \subseteq \Omega^\mathsf{T}; \mathfrak{J}^\mathsf{T}; (c ⧁ d) \;=\; (\Omega^\mathsf{T} ⧀ \Omega^\mathsf{T}); (c ⧁ d)$$

Proof: i) $\mathfrak{M}^\mathsf{T}; (c ⧁ d) = \mathtt{syq}(\varepsilon, (\varepsilon; \pi^\mathsf{T} \cap \varepsilon; \rho^\mathsf{T})); (\pi; c \cap \rho; d)$ by definition

$= \mathtt{syq}(\varepsilon, (\varepsilon; \pi^\mathsf{T} \cap \varepsilon; \rho^\mathsf{T}); (\pi; c \cap \rho; d))$ since $(\pi; c \cap \rho; d)$ is a point

$= \mathtt{syq}(\varepsilon, \varepsilon; \pi^\mathsf{T}; (\pi; c \cap \rho; d) \cap \varepsilon; \rho^\mathsf{T}; (\pi; c \cap \rho; d))$ again since $(\pi; c \cap \rho; d)$ is a point!

$= \mathtt{syq}(\varepsilon, \varepsilon; (c \cap \pi^\mathsf{T}; \rho; d) \cap \varepsilon; (\rho^\mathsf{T}; \pi; c \cap d))$

$= \mathtt{syq}(\varepsilon, \varepsilon; (c \cap \mathbb{T}) \cap \varepsilon; (\mathbb{T} \cap d))$

$= \mathtt{syq}(\varepsilon, \varepsilon; c \cap \varepsilon; d) =: s$, which is a point!

Now, we may continue

$$\Omega; \mathfrak{M}^\mathsf{T}; (c ⧁ d) = \Omega; s = \overline{\varepsilon^\mathsf{T}; \overline{\varepsilon}}; s = \overline{\varepsilon^\mathsf{T}; \overline{\varepsilon; s}} = \overline{\varepsilon^\mathsf{T}; \overline{\varepsilon; \mathtt{syq}(\varepsilon, \varepsilon; c \cap \varepsilon; d)}} = \overline{\varepsilon^\mathsf{T}; \overline{\varepsilon; c \cap \varepsilon; d}}$$

$$= \overline{\varepsilon^\mathsf{T}; (\overline{\varepsilon; c} \cup \overline{\overline{\varepsilon; d}})} = \overline{\varepsilon^\mathsf{T}; \overline{\varepsilon; c} \cup \varepsilon^\mathsf{T}; \overline{\overline{\varepsilon; d}}} = \overline{\varepsilon^\mathsf{T}; \overline{\varepsilon; c}} \cap \overline{\varepsilon^\mathsf{T}; \overline{\varepsilon; d}} = \overline{\varepsilon^\mathsf{T}; \overline{\varepsilon}}; c \cap \overline{\varepsilon^\mathsf{T}; \overline{\varepsilon}}; d = \Omega; c \cap \Omega; d \supseteq a$$

Short alternative proof:

$$\Omega; \mathfrak{M}^\mathsf{T}; (c ⧁ d) \;=\; (\Omega ⧀ \Omega); (c ⧁ d) \;= \Omega; c \cap \Omega; d \quad \text{Prop. 7.3.iii}$$

ii) is proved in a similar way. $\square$

One will understand Prop. 9.2.iv when interpreting it with cone intersection: Lower cone of a meet means intersecting the lower cones of the projections. Upper cone of the join is the intersection of the upper cones of the projections. Prop. 9.2.i resembles the De Morgan rule.

9.4 Proposition. Given any direct product with projections $\pi, \rho : X \times X \longrightarrow X$, and meet- or join-forming $\mathfrak{M}$, $\mathfrak{J}$,

i) the construct $p := \pi \cap \rho$ is univalent and surjective,

ii) meet-forming $\mathfrak{M}$ and join-forming $\mathfrak{J}$ are surjective mappings,

iii) concerning meet- and join-forming, $\mathfrak{J}$ **distributes** over $\mathfrak{M}$,

iv) meet-forming $\mathfrak{M}$ is a homomorphism and, even stronger, $(\Omega \otimes \Omega); \mathfrak{M} \;=\; \mathfrak{M}; \Omega$.

Proof: i) We use that the direct product encompasses every pair and that projections are surjective before applying the Dedekind formula

$$\mathbb{I} = \mathbb{T} \cap \mathbb{I} = \pi^\mathsf{T};\rho \cap \rho^\mathsf{T};\rho \subseteq (\pi^\mathsf{T} \cap \rho^\mathsf{T};\rho;\rho^\mathsf{T});(\rho \cap \pi;\rho^\mathsf{T};\rho) = (\pi^\mathsf{T} \cap \rho^\mathsf{T});(\rho \cap \pi)$$

ii) $\mathfrak{M} = \mathtt{syq}(\varepsilon;\pi^\mathsf{T} \cap \varepsilon;\rho^\mathsf{T}, \varepsilon) = \overline{(\varepsilon;\pi^\mathsf{T} \cap \varepsilon;\rho^\mathsf{T})^\mathsf{T};\varepsilon} \cap \overline{(\varepsilon;\pi^\mathsf{T} \cap \varepsilon;\rho^\mathsf{T})^\mathsf{T};\overline{\varepsilon}} =: A \cap B$

$$A = \overline{\overline{\pi;\varepsilon^\mathsf{T}};\varepsilon \cup \overline{\rho;\varepsilon^\mathsf{T}};\varepsilon} = \overline{\overline{\pi;\varepsilon^\mathsf{T}};\varepsilon} \cap \overline{\overline{\rho;\varepsilon^\mathsf{T}};\varepsilon} = \pi;\overline{\overline{\varepsilon^\mathsf{T}};\varepsilon} \cap \rho;\overline{\overline{\varepsilon^\mathsf{T}};\varepsilon} = \pi;\Omega^\mathsf{T} \cap \rho;\Omega^\mathsf{T}$$

$$B \supseteq \overline{(\pi;\varepsilon^\mathsf{T} \cup \rho;\varepsilon^\mathsf{T});\overline{\varepsilon}} = \overline{\pi;\varepsilon^\mathsf{T};\overline{\varepsilon} \cup \rho;\varepsilon^\mathsf{T};\overline{\varepsilon}} = \overline{\pi;\varepsilon^\mathsf{T};\overline{\varepsilon}} \cap \overline{\rho;\varepsilon^\mathsf{T};\overline{\varepsilon}} = \pi;\overline{\varepsilon^\mathsf{T};\overline{\varepsilon}} \cap \rho;\overline{\varepsilon^\mathsf{T};\overline{\varepsilon}} = \pi;\Omega \cap \rho;\Omega$$

$$\mathfrak{M} = A \cap B \supseteq (\pi;\Omega^\mathsf{T} \cap \rho;\Omega^\mathsf{T}) \cap (\pi;\Omega \cap \rho;\Omega) = \pi;(\Omega^\mathsf{T} \cap \Omega) \cap \rho;(\Omega^\mathsf{T} \cap \Omega) = \pi;\mathbb{I} \cap \rho;\mathbb{I} = \pi \cap \rho$$

The latter is surjective owing to (i). The proof for $\mathfrak{J}$ is rather similar.

iii) $((\pi \otimes \mathbb{I});\mathfrak{J} \olessthan (\rho \otimes \mathbb{I});\mathfrak{J});\mathfrak{M}$ where the first factor is a mapping

$= ((\pi \otimes \mathbb{I});\mathfrak{J} \olessthan (\rho \otimes \mathbb{I});\mathfrak{J});\mathtt{syq}((\varepsilon \olessthan \varepsilon), \varepsilon)$ definition of $\mathfrak{M}$

$= \mathtt{syq}((\varepsilon \olessthan \varepsilon);((\pi \otimes \mathbb{I});\mathfrak{J} \olessthan (\rho \otimes \mathbb{I});\mathfrak{J})^\mathsf{T}, \varepsilon)$

$= \mathtt{syq}((\varepsilon \olessthan \varepsilon);(\mathfrak{J}^\mathsf{T};(\pi^\mathsf{T} \otimes \mathbb{I}) \ogreaterthan \mathfrak{J}^\mathsf{T};(\rho^\mathsf{T} \otimes \mathbb{I})), \varepsilon)$ transposed

$= \mathtt{syq}(\varepsilon;\mathfrak{J}^\mathsf{T};(\pi^\mathsf{T} \otimes \mathbb{I}) \cap \varepsilon;\mathfrak{J}^\mathsf{T};(\rho^\mathsf{T} \otimes \mathbb{I}), \varepsilon)$ Prop. 7.3.i

$= \mathtt{syq}((\varepsilon;\pi^\mathsf{T} \cup \varepsilon;\rho^\mathsf{T});(\pi^\mathsf{T} \otimes \mathbb{I}) \cap (\varepsilon;\pi^\mathsf{T} \cup \varepsilon;\rho^\mathsf{T});(\rho^\mathsf{T} \otimes \mathbb{I}), \varepsilon)$ Prop. 9.1.iii

$= \mathtt{syq}((\varepsilon;\pi^\mathsf{T};(\pi^\mathsf{T} \otimes \mathbb{I}) \cup \varepsilon;\rho^\mathsf{T};(\pi^\mathsf{T} \otimes \mathbb{I})) \cap (\varepsilon;\pi^\mathsf{T};(\rho^\mathsf{T} \otimes \mathbb{I}) \cup \varepsilon;\rho^\mathsf{T};(\rho^\mathsf{T} \otimes \mathbb{I})), \varepsilon)$

$= \mathtt{syq}((\varepsilon;\pi^\mathsf{T};\pi'^\mathsf{T} \cup \varepsilon;\rho'^\mathsf{T}) \cap (\varepsilon;\rho^\mathsf{T};\pi'^\mathsf{T} \cup \varepsilon;\rho'^\mathsf{T}), \varepsilon)$

$= \mathtt{syq}((\varepsilon;\pi^\mathsf{T};\pi'^\mathsf{T} \cap \varepsilon;\rho^\mathsf{T};\pi'^\mathsf{T}) \cup \varepsilon;\rho'^\mathsf{T}, \varepsilon)$

$= \mathtt{syq}((\varepsilon;\pi^\mathsf{T} \cap \varepsilon;\rho^\mathsf{T});\pi'^\mathsf{T} \cup \varepsilon;\rho'^\mathsf{T}, \varepsilon)$

$= \mathtt{syq}([\varepsilon;\mathfrak{M}^\mathsf{T};\pi'^\mathsf{T} \cup \varepsilon;\rho'^\mathsf{T}], \varepsilon)$

$= \mathtt{syq}([\varepsilon;(\mathfrak{M}^\mathsf{T};\pi'^\mathsf{T} \cap \pi^\mathsf{T};\rho;\rho'^\mathsf{T}) \cup \varepsilon;(\rho^\mathsf{T};\pi;\mathfrak{M}^\mathsf{T};\pi'^\mathsf{T} \cap \rho'^\mathsf{T})], \varepsilon)$

$= \mathtt{syq}([\varepsilon;\pi^\mathsf{T};(\pi;\mathfrak{M}^\mathsf{T};\pi'^\mathsf{T} \cap \rho;\rho'^\mathsf{T}) \cup \varepsilon;\rho^\mathsf{T};(\pi;\mathfrak{M}^\mathsf{T};\pi'^\mathsf{T} \cap \rho;\rho'^\mathsf{T})], \varepsilon)$

$= \mathtt{syq}([\varepsilon;\pi^\mathsf{T} \cup \varepsilon;\rho^\mathsf{T}];(\pi;\mathfrak{M}^\mathsf{T};\pi'^\mathsf{T} \cap \rho;\rho'^\mathsf{T}), \varepsilon)$

$= \mathtt{syq}(\varepsilon;[\pi^\mathsf{T} \cup \rho^\mathsf{T}];(\mathfrak{M}^\mathsf{T} \otimes \mathbb{I}), \varepsilon)$

$= \mathtt{syq}(\varepsilon;[\pi \cup \rho]^\mathsf{T};(\mathfrak{M} \otimes \mathbb{I})^\mathsf{T}, \varepsilon)$

$= (\mathfrak{M} \otimes \mathbb{I});\mathtt{syq}(\varepsilon;[\pi \cup \rho]^\mathsf{T}, \varepsilon)$ since $(\mathfrak{M} \otimes \mathbb{I})$ is a mapping

$= (\mathfrak{M} \otimes \mathbb{I});\mathfrak{J}$ Prop. 9.1.i

iv) "$\subseteq$" follows with shunting $(\Omega \otimes \Omega);\mathfrak{M} \subseteq \mathfrak{M};\Omega \iff (\Omega \otimes \Omega) \subseteq \mathfrak{M};\Omega;\mathfrak{M}^\mathsf{T}$ from

$\mathfrak{M};\Omega;\mathfrak{M}^\mathsf{T} = \overline{\mathfrak{M};\varepsilon^\mathsf{T};\overline{\varepsilon;\mathfrak{M}^\mathsf{T}}} = \overline{(\varepsilon^\mathsf{T} \ogreaterthan \varepsilon^\mathsf{T});\overline{(\varepsilon \olessthan \varepsilon)}} = \overline{(\varepsilon \olessthan \varepsilon)^\mathsf{T};\overline{(\varepsilon \olessthan \varepsilon)}} = (\varepsilon \olessthan \varepsilon) \setminus (\varepsilon \olessthan \varepsilon)$

$\supseteq (\varepsilon\backslash\varepsilon \otimes \varepsilon\backslash\varepsilon)$ following Prop. 7.3.viii

$= (\Omega \otimes \Omega)$

The other direction "$\supseteq$" applies distributivity (iv):

$\mathfrak{M};\Omega = \pi'^\mathsf{T};\rho' \cap \mathfrak{M};\Omega$ π', ρ' form a direct product

$= \pi'^\mathsf{T};(\mathbb{T} \cap \rho') \cap \mathfrak{M};\Omega$ from now on using the abbreviation of Def. 8.8

$= \pi'^\mathsf{T};(\hat{\mathfrak{J}};\pi'^\mathsf{T};\rho' \cap \rho') \cap \mathfrak{M};\Omega$ $\hat{\mathfrak{J}}$ is total and π', ρ' form a direct product

$= \pi'^\mathsf{T};(\hat{\mathfrak{J}};\pi'^\mathsf{T} \cap \rho';\rho'^\mathsf{T});\rho' \cap \mathfrak{M};\Omega$ destroy and append

$= \left[\pi'^\mathsf{T};(\hat{\mathfrak{J}};\pi'^\mathsf{T} \cap \rho';\rho'^\mathsf{T}) \cap \mathfrak{M};\Omega;\rho'^\mathsf{T}\right];\rho'$ again destroy and append

$= \pi'^\mathsf{T};\left[\hat{\mathfrak{J}};\pi'^\mathsf{T} \cap \rho';\rho'^\mathsf{T} \cap \pi';\mathfrak{M};\Omega;\rho'^\mathsf{T}\right];\rho'$ again destroy and append

$= \pi'^\mathsf{T};\left[\hat{\mathfrak{J}};\pi'^\mathsf{T} \cap (\mathfrak{M} \otimes \mathbb{I});(\Omega \ogreaterthan \mathbb{I});\rho'^\mathsf{T}\right];\rho'$ due to Prop. 7.3.iii

$\subseteq \pi'^\mathsf{T};\left[\hat{\mathfrak{J}};\pi'^\mathsf{T} \cap (\mathfrak{M} \otimes \mathbb{I});\mathfrak{J};\rho'^\mathsf{T}\right];\rho'$ due to Prop. 9.2.viii

$\subseteq \pi'^\mathsf{T};\left[\hat{\mathfrak{J}};\pi'^\mathsf{T} \cap ((\pi \otimes \mathbb{I});\mathfrak{J} \olessthan (\rho \otimes \mathbb{I});\mathfrak{J});\mathfrak{M};\rho'^\mathsf{T}\right];\rho'$ due to Def. 8.8, (iv)

$\subseteq \pi'^\mathsf{T};\left[\hat{\mathfrak{J}};\pi'^\mathsf{T} \cap \hat{\mathfrak{J}};\mathfrak{M};\rho'^\mathsf{T}\right];\rho'$

$\subseteq \pi'^\mathsf{T};\hat{\mathfrak{J}};\left[\pi'^\mathsf{T} \cap \mathfrak{M};\rho'^\mathsf{T}\right];\rho'$

$$
\begin{aligned}
&\subseteq \pi'^{\mathsf{T}};\hat{\mathfrak{J}};\left[\pi'^{\mathsf{T}};\rho' \cap \mathfrak{M}\right]\\
&\subseteq \pi'^{\mathsf{T}};\hat{\mathfrak{J}};\mathfrak{M}\\
&= \pi'^{\mathsf{T}};\big(\,(\pi \otimes \mathbb{I});\mathfrak{J} \olessthan (\rho \otimes \mathbb{I});\mathfrak{J}\,\big);\mathfrak{M} \quad \text{expanded}\\
&\subseteq \big(\pi'^{\mathsf{T}};(\pi \otimes \mathbb{I});\mathfrak{J} \olessthan \pi'^{\mathsf{T}};(\rho \otimes \mathbb{I});\mathfrak{J}\,\big);\mathfrak{M}\\
&= \big(\pi;\pi^{\mathsf{T}};\mathfrak{J} \olessthan \rho;\pi^{\mathsf{T}};\mathfrak{J}\,\big);\mathfrak{M}\\
&= (\pi;\Omega \olessthan \rho;\Omega);\mathfrak{M}\\
&= (\Omega \otimes \Omega);\mathfrak{M} \qquad \square
\end{aligned}
$$

When we proceed according to (iii) from a pair of sets to a set of possibly bigger ones and form their meet, we might also first form the meet and then increase.

	{}	{a}	{b}	{a,b}	{c}	{a,c}	{b,c}	{a,b,c}	{d}	{a,d}	{b,d}	{a,b,d}	{c,d}	{a,c,d}	{b,c,d}	{a,b,c,d}
({},{})	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({a},{})	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({},{a})	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({b},{})	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
({a},{a})	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({},{b})	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
({a,b},{})	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
({b},{a})	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
({a},{b})	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
({},{a,b})	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
({c},{})	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
({a,b},{a})	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
({b},{b})	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
({a},{a,b})	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
({},{c})	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
({a,c},{})	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
({c},{a})	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
({a,b},{b})	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
({b},{a,b})	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
({a},{c})	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
({},{a,c})	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
({b,c},{})	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
({a,c},{a})	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
({c},{b})	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
({a,b},{a,b})	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
({b},{c})	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
({a},{a,c})	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
({},{b,c})	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
({a,b,c},{})	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
({b,c},{a})	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
({a,c},{b})	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
({c},{a,b})	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
({a,b},{c})	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
({b},{a,c})	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
({a},{b,c})	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
({},{a,b,c})	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
({d},{})	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0

	{}	{a}	{b}	{a,b}	{c}	{a,c}	{b,c}	{a,b,c}	{d}	{a,d}	{b,d}	{a,b,d}	{c,d}	{a,c,d}	{b,c,d}	{a,b,c,d}
({},{})	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({a},{})	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({},{a})	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({b},{})	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({a},{a})	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({},{b})	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({a,b},{})	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({b},{a})	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({a},{b})	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({},{a,b})	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({c},{})	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({a,b},{a})	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({b},{b})	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
({a},{a,b})	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({},{c})	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({a,c},{})	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({c},{a})	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({a,b},{b})	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
({b},{a,b})	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
({a},{c})	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({},{a,c})	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({b,c},{})	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({a,c},{a})	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({c},{b})	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({a,b},{a,b})	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
({b},{c})	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({a},{a,c})	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({},{b,c})	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({a,b,c},{})	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({b,c},{a})	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({a,c},{b})	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({c},{a,b})	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({a,b},{c})	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({b},{a,c})	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({a},{b,c})	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({},{a,b,c})	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
({d},{})	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Fig. 9.5 The initial ones of 256 rows of the relations $\mathfrak{J}, \mathfrak{M} : \mathbf{2}^X \times \mathbf{2}^X \longrightarrow \mathbf{2}^X$

One will identify the commutative law in (ii,iii), where it is expressed that the collection of results doesn't change when starting from the first as opposed to the second component. The other laws may be found later in Prop. 9.5.

	{}	{a}	{b}	{a,b}	{c}	{a,c}	{b,c}	abc	{d}	{a,d}	{b,d}	abd	{c,d}	acd	bcd	all
{}	{}	{a}	{b}	{a,b}	{c}	{a,c}	{b,c}	abc	{d}	{a,d}	{b,d}	abd	{c,d}	acd	bcd	all
{a}	{a}	{a}	{a,b}	{a,b}	{a,c}	{a,c}	abc	abc	{a,d}	{a,d}	abd	abd	acd	acd	all	all
{b}	{b}	{a,b}	{b}	{a,b}	{b,c}	abc	{b,c}	abc	{b,d}	abd	{b,d}	abd	bcd	all	bcd	all
{a,b}	{a,b}	{a,b}	{a,b}	{a,b}	abc	abc	abc	abc	abd	abd	abd	abd	all	all	all	all
{c}	{c}	{a,c}	{b,c}	abc	{c}	{a,c}	{b,c}	abc	{c,d}	acd	bcd	all	{c,d}	acd	bcd	all
{a,c}	{a,c}	{a,c}	abc	abc	{a,c}	{a,c}	abc	abc	acd	acd	all	all	acd	acd	all	all
{b,c}	{b,c}	abc	{b,c}	abc	{b,c}	abc	{b,c}	abc	bcd	all	bcd	all	bcd	all	bcd	all
abc	abc	abc	abc	abc	abc	abc	abc	abc	all	all	all	all	all	all	all	all
{d}	{d}	{a,d}	{b,d}	abd	{c,d}	acd	bcd	all	{d}	{a,d}	{b,d}	abd	{c,d}	acd	bcd	all
{a,d}	{a,d}	{a,d}	abd	abd	acd	acd	all	all	{a,d}	{a,d}	abd	abd	acd	acd	all	all
{b,d}	{b,d}	abd	{b,d}	abd	bcd	all	bcd	all	{b,d}	abd	{b,d}	abd	bcd	all	bcd	all
abd	abd	abd	abd	abd	all	all	all	all	abd	abd	abd	acd	all	all	all	all
{c,d}	{c,d}	acd	bcd	all	{c,d}	acd	bcd	all	{c,d}	acd	bcd	all	{c,d}	acd	bcd	all
acd	acd	acd	all	all	acd	acd	all	all	acd	acd	all	all	acd	acd	all	all
bcd	bcd	all	bcd	all	bcd	all	bcd	all	bcd	all	bcd	all	bcd	all	bcd	all
{a,b,c,d}	all	all	all	all	all	all	all	all	all	all	all	all	all	all	all	all

Fig. 9.6 $\mathfrak{J}$ as function table $\mathfrak{J} \in \left[\mathbf{2}^X\right]^{\mathbf{2}^X \times \mathbf{2}^X}$; abbreviated notation for 3- and 4-element sets

9.5 Proposition. $\mathfrak{J}$, $\mathfrak{M}$ satisfy

i) $\left[\pi^\mathsf{T} \cap \rho^\mathsf{T}; \mathfrak{M}; \rho^\mathsf{T}\right]; \mathfrak{J} = \mathbb{I}$, $\quad \left[\pi^\mathsf{T} \cap \rho^\mathsf{T}; \mathfrak{J}; \rho^\mathsf{T}\right]; \mathfrak{M} = \mathbb{I}$, $\quad$ i.e., the absorption laws

ii) $(\mathfrak{M} \otimes \mathbb{I}); \mathfrak{M} = T; (\mathbb{I} \otimes \mathfrak{M}); \mathfrak{M}$ $\quad$ i.e., the associative law, where

$T : (X \times X) \times X \longrightarrow X \times (X \times X)$ is the brace rearrangement bijection of Def. 8.2.

Proof: i) We start the proof of "$\subseteq$" with Prop. 9.2.i, Prop. 9.1.i and shunting.

$$\left[\pi^\mathsf{T} \cap \rho^\mathsf{T}; \mathfrak{M}; \rho^\mathsf{T}\right]; \mathfrak{J} = \left[\pi^\mathsf{T} \cap \Omega^\mathsf{T}; \rho^\mathsf{T}\right]; \mathfrak{J} \subseteq \mathbb{I} \quad \Longleftrightarrow \quad \pi^\mathsf{T} \cap \Omega^\mathsf{T}; \rho^\mathsf{T} \subseteq \mathfrak{J}^\mathsf{T}$$

$$\iff \quad \overline{\mathfrak{J}} \subseteq \overline{\pi} \cup \overline{\rho;\Omega} \quad \iff \quad \overline{\varepsilon;\pi^{\mathsf{T}} \cup \varepsilon;\rho^{\mathsf{T}}}^{\mathsf{T}};\varepsilon \cup \left[\varepsilon;\pi^{\mathsf{T}} \cup \varepsilon;\rho^{\mathsf{T}}\right]^{\mathsf{T}};\overline{\varepsilon} \subseteq \overline{\pi} \cup \rho;\varepsilon^{\mathsf{T}};\overline{\varepsilon}$$

The first term is contained in $\overline{\pi}$, because

$$(\pi;\overline{\varepsilon}^{\mathsf{T}} \cap \rho;\overline{\varepsilon}^{\mathsf{T}});\varepsilon \subseteq \overline{\pi;\varepsilon^{\mathsf{T}}};\varepsilon \subseteq \overline{\pi}$$

The second term is also contained in $\overline{\pi}$, owing to univalency of π

$$\pi;\varepsilon^{\mathsf{T}};\overline{\varepsilon} \subseteq \overline{\pi} \quad \iff \quad \varepsilon;\pi^{\mathsf{T}};\pi \subseteq \varepsilon$$

Finally, the third term is equal to the right-most one. This was the proof of containment only; but this suffices because the total (see Prop. 9.4.ii) term $\left[\pi^{\mathsf{T}} \cap \rho^{\mathsf{T}};\mathfrak{M};\rho^{\mathsf{T}}\right];\mathfrak{J}$ contained in the univalent $\mathbb{I}$, so that both must be equal.

ii) $T;(\mathbb{I} \otimes \mathfrak{M});\mathfrak{M} = T;(\mathbb{I} \otimes \mathfrak{M});\mathtt{syq}((\varepsilon \olessthan \varepsilon), \varepsilon)$ Prop. 9.1.ii
$= T;\mathtt{syq}((\varepsilon \olessthan \varepsilon);(\mathbb{I} \otimes \mathfrak{M})^{\mathsf{T}}, \varepsilon)$ since $(\mathbb{I} \otimes \mathfrak{M})$ is a mapping
$= T;\mathtt{syq}((\varepsilon \olessthan \varepsilon);(\mathbb{I} \otimes \mathfrak{M}^{\mathsf{T}}), \varepsilon)$ transposed
$= T;\mathtt{syq}((\varepsilon \olessthan \varepsilon;\mathfrak{M}^{\mathsf{T}}), \varepsilon)$ Prop. 7.3.iii
$= T;\mathtt{syq}((\varepsilon \olessthan (\varepsilon \olessthan \varepsilon)), \varepsilon)$ Prop. 9.1.iv
$= (\pi';\pi \olessthan (\rho \otimes \mathbb{I}));\mathtt{syq}((\varepsilon \olessthan (\varepsilon \olessthan \varepsilon)), \varepsilon)$ expanding T according to Def. 8.2.iii
$= \mathtt{syq}((\varepsilon \olessthan (\varepsilon \olessthan \varepsilon));(\pi';\pi \olessthan (\rho \otimes \mathbb{I}))^{\mathsf{T}}, \varepsilon)$ T is a map
$= \mathtt{syq}((\varepsilon \olessthan (\varepsilon \olessthan \varepsilon));(\pi^{\mathsf{T}};\pi'^{\mathsf{T}} \ogreaterthan (\rho^{\mathsf{T}} \otimes \mathbb{I})), \varepsilon)$ transposed
$= \mathtt{syq}(\varepsilon;\pi^{\mathsf{T}};\pi'^{\mathsf{T}} \cap (\varepsilon \olessthan \varepsilon);(\rho^{\mathsf{T}} \otimes \mathbb{I}), \varepsilon)$ Prop. 7.4
$= \mathtt{syq}(\varepsilon;\pi^{\mathsf{T}};\pi'^{\mathsf{T}} \cap (\varepsilon;\rho^{\mathsf{T}} \olessthan \varepsilon), \varepsilon)$ Prop. 7.3.iii
$= \mathtt{syq}(\varepsilon;\pi^{\mathsf{T}};\pi'^{\mathsf{T}} \cap \varepsilon;\rho^{\mathsf{T}};\pi'^{\mathsf{T}} \cap \varepsilon;\rho'^{\mathsf{T}}, \varepsilon)$
$= \mathtt{syq}((\varepsilon;\pi^{\mathsf{T}} \cap \varepsilon;\rho^{\mathsf{T}});\pi'^{\mathsf{T}} \cap \varepsilon;\rho'^{\mathsf{T}}, \varepsilon)$
$= \mathtt{syq}((\varepsilon \olessthan \varepsilon);\pi'^{\mathsf{T}} \cap \varepsilon;\rho'^{\mathsf{T}}, \varepsilon)$
$= \mathtt{syq}(((\varepsilon \olessthan \varepsilon) \olessthan \varepsilon), \varepsilon)$
$= \mathtt{syq}((\varepsilon;\mathfrak{M}^{\mathsf{T}} \olessthan \varepsilon), \varepsilon)$ Prop. 9.1.iv
$= \mathtt{syq}((\varepsilon \olessthan \varepsilon);(\mathfrak{M}^{\mathsf{T}} \otimes \mathbb{I}), \varepsilon)$
$= \mathtt{syq}((\varepsilon \olessthan \varepsilon);(\mathfrak{M} \otimes \mathbb{I})^{\mathsf{T}}, \varepsilon)$
$= (\mathfrak{M} \otimes \mathbb{I});\mathtt{syq}((\varepsilon \olessthan \varepsilon), \varepsilon)$
$= (\mathfrak{M} \otimes \mathbb{I});\mathfrak{M}$ □

10 Concluding Remarks

These additions have already been broadly applied, not least in studies of relational topology. The relational language TituRel reflecting all these ideas in functional programming style has made it possible to successfully explore discrete topologies, concepts of nearness, proximity that have been studied by logicians.

These investigations further support our firm creed: Mankind seems hardly capable of handling intellectually more than *linear* situations!

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Index