A Priori Error Estimates for an Optimal Control Problem Governed by a Variational Inequality of the Second Kind

Christian Meyer[1,*] and **Monika Weymuth**[2,**]

[1] Technische Universität Dortmund, Fakultät für Mathematik (LS X), DE-44221 Dortmund
[2] Universität der Bundeswehr München, Werner-Heisenberg-Weg 39, DE-85577 Neubiberg

We consider an optimal control problem governed by an elliptic variational inequality of the second kind. The problem is discretized by linear finite elements for the state and a variational discrete approach for the control. We derive nearly optimal a priori error estimates based on L^∞-error estimates for the variational inequality and a quadratic growth condition. Our error analysis yields a convergence rate of order $1 - \epsilon$ for the L^2-norm of the control.

1 Introduction

Variational inequalities (VIs) of the second kind are an important tool for modelling different processes in mechanics (see [6]). In many applications the optimization of these processes is of interest which leads to an optimal control problem. Such problems have to be solved numerically. Hence, discretization error estimates are of great importance in practice.

There are a lot of contributions in the field of a priori error analysis for optimal control problems governed by PDEs. However, to the authors' best knowledge, there is only one paper [5] deriving quantitative error estimates for a finite element discretization of an optimal control problem governed by a VI, namely in the special case of the obstacle problem.

2 The Optimal Control Problem

We consider the following optimal control problem governed by a VI of the second kind:

$$\min_{y \in H_0^1(\Omega), u \in L^2(\Omega)} J(y, u) := \frac{1}{2}\|y - y_d\|_{L^2(\Omega)}^2 + \frac{\nu}{2}\|u\|_{L^2(\Omega)}^2 \tag{1a}$$

$$\text{s.t.} \quad \int_\Omega \nabla y \cdot \nabla(v - y)\, \mathrm{d}x + \|v\|_{L^1(\Omega)} - \|y\|_{L^1(\Omega)} \geq \langle u, v - y \rangle \quad \forall v \in H_0^1(\Omega) \tag{1b}$$

$\Omega \subset \mathbb{R}^d$ $(d = 1, 2)$ is a bounded domain with $C^{1,1}$-boundary, $y_d \in L^2(\Omega)$ is the desired state and $\nu > 0$ a fixed real number.

Standard arguments yield that (1b) has a unique solution for every $u \in H^{-1}(\Omega)$ (see [1]). The associated solution operator $S : H^{-1}(\Omega) \to H_0^1(\Omega)$ mapping u to y is globally Lipschitz continuous. Moreover existence of a globally optimal solution of problem (1) follows by standard arguments (cf. [1]). We state the following result concerning the regularity of local optima.

Proposition 2.1 *Every locally optimal solution satisfies* $\overline{u} \in H^1(\Omega)$.

3 Discretization

We discretize (1) with linear finite elements and introduce a family of meshes $\{\mathcal{T}_h\}$ with mesh size h. The mesh $\mathcal{T}_h$ consists of open simplices (intervals in 1D, triangles in 2D) and is assumed to be shape-regular and quasi-uniform. Since the boundary of Ω is of class $C^{1,1}$, the discrete domain $\Omega_h := \bigcup_{T \in \mathcal{T}_h} T$ differs from Ω. We assume that $\Omega_h \subseteq \Omega$ and $|\Omega \setminus \bigcup_{T \in \mathcal{T}_h} T| \lesssim h^2$.

The variational discretization of (1) is given by

$$\min_{y_h \in V_h, u \in L^2(\Omega)} J(y_h, u) := \frac{1}{2}\|y_h - y_d\|_{L^2(\Omega)}^2 + \frac{\nu}{2}\|u\|_{L^2(\Omega)}^2 \tag{2a}$$

$$\text{s.t.} \quad \int_{\Omega_h} \nabla y_h \cdot \nabla(v_h - y_h)\, \mathrm{d}x + \|v_h\|_{L^1(\Omega_h)} - \|y_h\|_{L^1(\Omega_h)} \geq \langle u, v_h - y_h \rangle \quad \forall v_h \in V_h \tag{2b}$$

with $V_h := \{v_h \in H_0^1(\Omega_h) : v_h|_T \in \mathbb{P}_1(T)\ \forall T \in \mathcal{T}_h\}$.

---

* cmeyer@math.tu-dortmund.de
** Corresponding author: e-mail monika.weymuth@unibw.de, phone +49 89 6004 3859, fax +49 89 6004 4136

Note that the control u is not discretized. However, due to the following result it suffices to restrict the controls to the set V_h in order to obtain a fully discrete optimization problem.

Proposition 3.1 *If $\overline{u}_h$ is a local optimal solution of (2), then $\overline{u}_h \in V_h$.*

4   Strong Stationarity

Since the control-to-state operator S as defined in Section 2 is in general not Gâteaux differentiable, the standard adjoint approach for the derivation of first order necessary optimality conditions cannot be applied. There are essentially two different alternative approaches. The first is based on regularization. This technique is applied in [3]. In this paper a family of regularized optimal control problems is introduced and for each regularized problem optimality conditions are derived. Finally the optimality conditions for the original problem are obtained as a limit of the regularized ones. However, there is a loss of information by the passage to the limit and the optimality conditions are not rigorous enough for our purposes. Therefore we follow the second approach which is based on differentiability properties of the control-to-state operator S. In [4] the directional differentiability of S is investigated and strong stationarity conditions are derived. The differentiability results of S are generalized in [2]. In order to obtain strong stationarity conditions for our problem we use the differentiability result of [2] and combine it with the technique introduced in [4].

5   Error Analysis

The derivation of nearly optimal a priori error estimates is based on the error analysis in [5]. Moreover a key tool for the derivation of our convergence rates is the following result which can be found in [7].

Theorem 5.1 (Nochetto 1988) *If $u \in L^p(\Omega)$, $p \in (1, \infty)$, then there exists a constant $C > 0$ independent of h such that*

$$\|y - y_h\|_{L^\infty(\Omega)} \leq C |\log(h)| h^{2-\frac{d}{p}} (\|u\|_{L^p(\Omega)} + 1).$$

If $u \in L^\infty(\Omega)$, then there exists a constant $C > 0$ independent of h such that

$$\|y - y_h\|_{L^\infty(\Omega)} \leq C (h |\log(h)|)^2 (\|u\|_{L^\infty(\Omega)} + 1).$$

In the following let $\overline{u} \in L^2(\Omega)$ be a fixed local optimum.

Assumption 5.2 (quadratic growth condition) *There are ϵ, $\delta > 0$ such that*

$$J(S(\overline{u}), \overline{u}) \leq J(S(u), u) - \delta \|u - \overline{u}\|_{L^2(\Omega)}^2 \quad \forall u \in L^2(\Omega) \text{ s.t. } \|u - \overline{u}\|_{L^2(\Omega)} < \epsilon.$$

Lemma 5.3 *Suppose that $\overline{u}$ satisfies Assumption 5.2. Then there is a sequence $\{\overline{u}_h\}$ of locally optimal solutions to (2) with $\overline{u}_h \to \overline{u}$ in $L^2(\Omega)$ as $h \to 0$. Moreover the sequence $\{u_h\}$ is uniformly bounded in $H^1(\Omega)$.*

Based on Theorem 5.1, Lemma 5.3 and the continuous embeddings $H^1(\Omega) \hookrightarrow C(\overline{\Omega})$ in 1D, $H^1(\Omega) \hookrightarrow L^p(\Omega) \, \forall p < \infty$ in 2D we obtain the following a priori error estimates for the control.

Theorem 5.4 *Let $\Omega \subset \mathbb{R}$. If $\overline{u}$ satisfies Assumption 5.2, then there is a constant $C > 0$ such that, for h sufficiently small,*

$$\|\overline{u} - \overline{u}_h\|_{L^2(\Omega)} \leq C h |\log(h)|.$$

Theorem 5.5 *Let $\Omega \subset \mathbb{R}^2$ be sufficiently regular. If $\overline{u}$ satisfies Assumption 5.2, then there is a constant $C > 0$ such that, for h sufficiently small,*

$$\|\overline{u} - \overline{u}_h\|_{L^2(\Omega)} \leq C \sqrt{h^{2-\frac{d}{p}} |\log h|}.$$

References

[1] V. Barbu, Optimal Control of Variational Inequalities (Pitman, Boston, 1984).
[2] C. Christof, and C. Meyer, Set-valued Var. Anal. (2018).
[3] J. C. De los Reyes, SIAM J. Control Optim. **49**, 1629-1658 (2011).
[4] J. C. De los Reyes, and C. Meyer, J. Optim. Theory Appl. **168**, 375-409 (2016).
[5] C. Meyer, and O. Thoma, SIAM J. Numer. Anal. **51**, 605-628 (2013).
[6] J. C. De los Reyes, GAMM-Mitt. **40**, 268-286 (2017).
[7] R. H. Nochetto, Numer. Math. **54**, 243-255 (1988).