Isogeometric Analysis of bridge structures: State of the art and potential advantages

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Abstract

The Isogeometric Analysis is a novel method for the numerical solution of boundary value problems of different types. Since its introduction in 2005, it has successfully been applied to different problems of structural mechanics, among others. It offers significant advantages compared to the classical Finite Element Method. Nevertheless, this method is barely used in practice nowadays. In this contribution we first summarise the fundamentals of Isogeometric Analysis using Non-uniform rational B-Spline basis functions for both the geometric description of a structure and the numerical approximation of a boundary value problem solution field (e.g. deformations). We then offer an overview of recent applications of Isogeometric Analysis in the context of structural engineering with special focus on bridge constructions. Finally, we highlight potential advantages of applying Isogeometric Analysis in future bridge design. In this contribution, we focus on the benefits of a CAD-integrated, parametric design and analysis process, the advantages of geometric reduction of three-dimensional systems as well as the consistent data exchange in a digital workflow. These advantages are demonstrated using the example of a bridge superstructure.

Keywords: IGA; NURBS; bridges; design; digitalisation

1 Introduction

The Isogeometric Analysis (IGA) has been introduced in 2005 by Hughes et al. [1] as a computational engineering approach that integrates CAD and numerical analysis using NURBS or other CAD-based geometries as the basis for finite element analysis. This ensures a smooth and accurate representation of complex geometries, leading to higher accuracy, reduced remeshing efforts, and improved design flexibility compared to traditional FEA [2]. Numerous research studies have demonstrated the improved performance of IGA in structural mechanics, fluid dynamics, electromagnetics, acoustics, vibrations, biomechanics, and multiphysics problems, and have shown its applicability in various engineering fields, from aerospace and automotive to medical devices and bioengineering research. In some of these fields, e.g. automotive, IGA has already moved from academic research to practical application in the industry. So far, less attention has been given to the application of IGA in civil and structural engineering problems although it bears great potential also there.

In the field of bridge design and analysis, IGA has so far only been applied on a scientific basis for special topics. Helgedagsrud et al. performed aerodynamic simulations of turbulent flows at the Hardanger Bridge in Norway using IGA. A comparison of the calculated solutions to results of wind tunnel experiments showed that the aerodynamic forces on the bridge superstructure were numerically approximated with a very good accuracy and the performance of the calculation using IGA was significantly better than using FEM [3]. Van Do et. al. investigated the dynamic behaviour of a Euler-Bernoulli beam structure under loading by vehicles with moving axle loads. Different surface roughnesses were also taken into account. They found that the application of IGA in this area leads to reduced simulation time and costs [4]. Zhang et al. performed topology optimisation on a prestressed beam using IGA [5]. Tsiptsis and Sapountzakis studied the generalized warping and distortional behaviour of beams with open and closed cross-sections. They applied IGA to reduce the computational cost and exactly represent the curved geometry of the analysed bridge structure [6].

In this paper we show that there is great potential for the practical application of IGA in bridge design and analysis. First, in Section 2 the basics of geometric modelling using NURBS, and in Section 3, the basics of IGA are summarised. In the following, exemplary advantages of using NURBS and IGA in a CAD-integrated, parametric design and analysis of bridges (Section 4.1), for a dimensional reduction process (Section 4.1) and for the consistent data exchange in a digital workflow (Section 4.3) are illustrated.

2 Geometric modelling using NURBS

2.1 General

Geometric modelling of curves, surfaces and solids using NURBS is considered an industry standard in the field of computer graphics [7]. Using NURBS, it is possible to describe and represent three-dimensional geometric objects, both as analytical standard shapes and in free form. Even complex geometric shapes can be created by joining less complex pieces, called patches. Furthermore, using NURBS, geometric objects that are described by means of rational functions can be represented [7-9]. NURBS represent a generalised form of so-called Bézier splines (B-splines) and belong to the category of approximating methods for the mathematical description of geometric objects [10]. The basics of geometric modelling using NURBS are explained below. Relevant parameters and variables are introduced in Table 1. Further background information, a detailed derivation of the underlying mathematical functions and hints for the implementation in software applications can be found in [7] and [9].

Table 1: Parameters and variables for the definition of NURBS entities [7-9; 11].

Parameter / Variable	Description	Set of numbers
x, y, z	Cartesian coordinates	$\mathbb{R}$
P	NURBS control point	$\mathbb{R}^3$
w	Weight of a NURBS control point	$\mathbb{R}_+$
C, S, V	Point on a NURBS curve, surface or solid	$\mathbb{R}^3$
ξ, η, ζ	Parameter values	$\mathbb{R}_+$
nk	Number of knots in a knot vector	$\mathbb{Z}_+$
Ξ, H, Z	Knot vectors	${\mathbb{R}_+}^{nk}$
p, q, r	Polynomial degrees of the NURBS basis functions	$\mathbb{Z}_+$
n, m, l	Number of the NURBS basis functions	$\mathbb{Z}_+$
$i, j, k, \hat{i}, \hat{j}, \hat{k}$	Control variables	${\mathbb{Z}_+}^n, {\mathbb{Z}_+}^m, {\mathbb{Z}_+}^l$
N, M, L	B-Spline basis functions	$\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^l$
R	NURBS basis functions	$\mathbb{R}^{n \cdot m \cdot l}$

2.2 NURBS basis functions

In NURBS based geometric modelling and IGA, basis functions **R** are applied to approximate both the geometric object itself and BVP solution fields [7; 8]. To generate these basis functions, at first a one-dimensional (curves), two-dimensions (surfaces) or three-dimensional (solids) parameter space, enclosing the geometry, must be introduced. This parameter space is described by knot vectors **Ξ**, **H** and **Z** according to Eq. (1). Knot vectors are defined in ascending order and have repeating knots at their start and end. They usually start at 0 and end at 1 [7]. The number of knots is defined by the chosen grade of the basis functions p, q and r.

$$\boldsymbol{\Xi} = \left\{\underbrace{0, \ldots, 0}_{p+1}, \xi_{p+1}, \ldots, \xi_{n-p-1}, \underbrace{1, \ldots, 1}_{p+1}\right\} \tag{1}$$

(Analogously valid for **H** and **Z** with q, r instead of p and m, l instead of n)

Once the parameter space has been defined, B-Spline basis functions are calculated using the Cox-de Boor recursion formula, see Eq. (2) and Eq. (3) [7]. NURBS basis functions are then defined by a projective transformation according to Eq. (4) [7; 8].

$$N_{i,0}(\xi) = \begin{cases} 0, & \text{if } \xi \in [\xi_i, \xi_{i+p+1}[\\ 1, & \text{otherwise} \end{cases} \tag{2}$$

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} \cdot N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} \cdot N_{i+1,p-1}(\xi) \tag{3}$$

(Analogously valid for M and L with η, ζ instead of ξ and q, r instead of p and m, l instead of n)

$$R_{i,j,k,p,q,r}(\xi,\eta,\zeta) = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l} \frac{N_{i,p}(\xi)\, M_{j,q}(\eta)\, L_{k,r}(\zeta)\, w_{i,j,k}}{\sum_{\hat{i}=1}^{n} \sum_{\hat{j}=1}^{m} \sum_{\hat{k}=1}^{l} N_{\hat{i},p}(\xi)\, M_{\hat{j},q}(\eta)\, L_{\hat{k},r}(\zeta)\; w_{\hat{i},\hat{j},\hat{k}}} \tag{4}$$

2.3 NURBS curves, surfaces and solids

Spatial curves $\mathbf{C}(\xi)$ and three-dimensional surfaces $\mathbf{S}(\xi,\eta)$ and solids $\mathbf{V}(\xi,\eta,\zeta)$ are described by Eq. (5), Eq. (6) and Eq. (7), respectively [8; 9]. To compute the cartesian coordinates of an individual point on a curve, surface or solid, the tensor product of NURBS basis functions $\mathbf{R}$ and predefined NURBS control points $\mathbf{P}$ must be calculated [7; 9].

$$\mathbf{C}(\xi)=\sum_{i=1}^{n} R_{i,p}(\xi)\ \mathbf{P}_i \tag{5}$$

$$\mathbf{S}(\xi,\eta)=\sum_{i=1}^{n}\sum_{j=1}^{m} R_{i,j,p,q}(\xi,\eta)\ \mathbf{P}_{i,j} \tag{6}$$

$$\mathbf{V}(\xi,\eta,\zeta)=\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} R_{i,j,k,p,q,r}(\xi,\eta,\zeta)\ \mathbf{P}_{i,j,k} \tag{7}$$

2.4 Example

The description of a NURBS surface is shown in Figure 1 as an example. First, the control-net (matrix of non-interpolatory control points) is defined, see black markings in Figure 1d). Then a point in parameter space is chosen (here: ξ = 0,25 and η = 0,75) and the associated NURBS basis function values are calculated, see star-symbols in Figure 1a), Figure 1b) and Figure 1c). Finally, the corresponding point in physical space (cartesian coordinate system) is computed using Eq. (6), see star-symbol in Figure 1d) [11]. This procedure is applied analogously for the description of curves and solids.

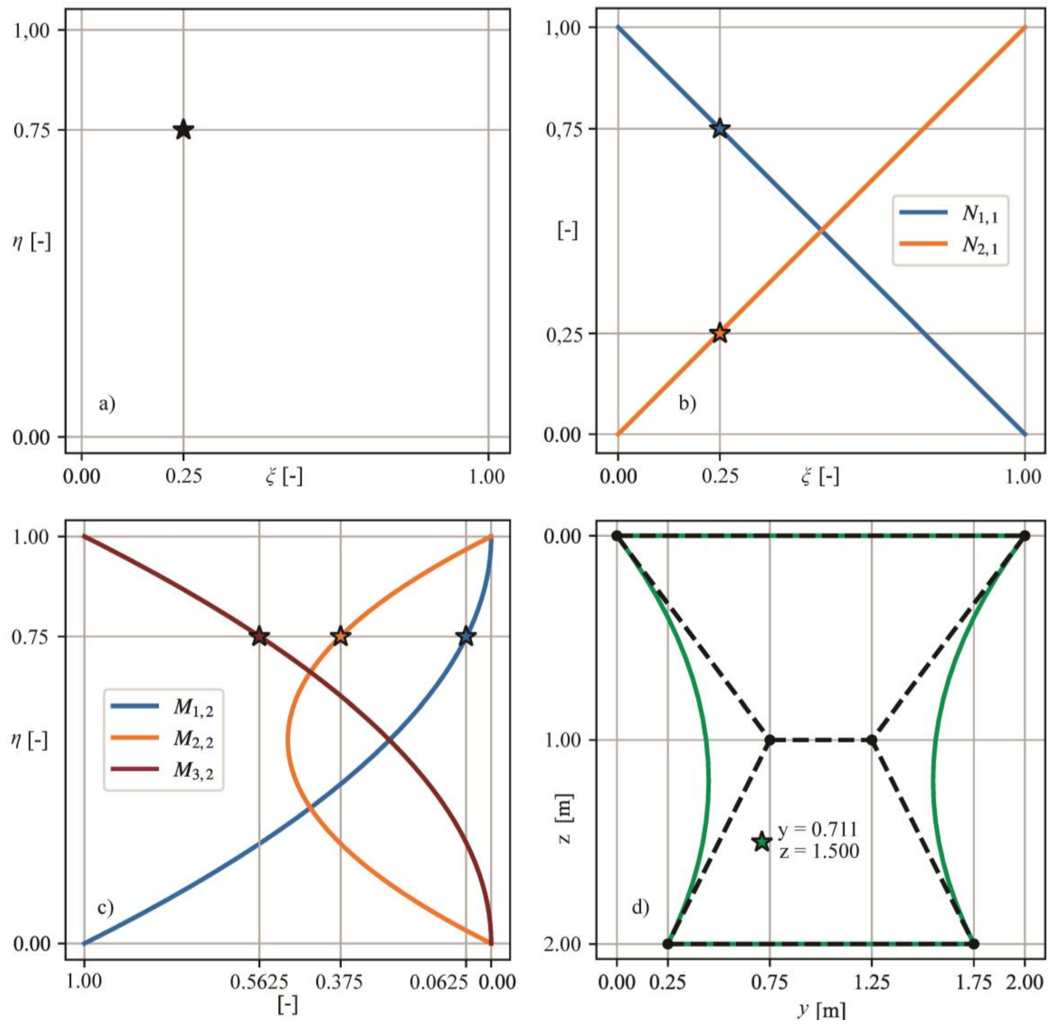

Figure 1: Representation of a NURBS surface with p = 1 and q = 2; a) Parameter space; b) NURBS basis functions for ξ; c) NURBS basis functions for η; d) Physical space with geometry of the surface [11].

3 Isogeometric Analysis

Within the framework of this paper, the term IGA refers to the numerical method of Bubnov-Galerkin, which is also used in "classical" FEM [12]. The aim of IGA is to approximatively solve differential equations which describe a solution field over a domain defined by NURBS entities, e.g. deformations of a structure. First, a weak form of the BVP is constructed using the calculus of variations. The trial solutions *S* and the weighting functions *V* generated in this process do have the NURBS basis functions of the geometric description as their basis and are partitioned into finite subsets S^h and V^h of the entire solution space (Bubnov-Galerkin method) [8].

These subsets are called elements. Different element types for the numerical analysis using IGA, like the Euler-Bernoulli beam or the Kirchhoff-Love shell, have since been developed and can be applied e.g. to the linear and nonlinear analysis of structures [13; 14]. Elements E are defined in the parameter space by inserting knots into the knot vector(s) of the NURBS description, see Eq. (8) as an example. This leads to the fact that in IGA, the subdivision of the geometry into elements is not accompanied by a geometric discretisation process and the exact geometry is always preserved. If structures are to be calculated whose shape cannot be described by a single NURBS entity, several NURBS curves, surfaces or solids can be connected. In IGA, these connected NURBS entities are called patches [8].

$$\boldsymbol{\Xi} = \left\{ 0,\ \underbrace{0,\ 0.5,\ 1}_{[\mathrm{E}_1 \mid \mathrm{E}_2]},1 \right\} \tag{8}$$

Likewise in FEM, stiffness matrices and load vectors are then generated for each element and assembled to their global counterparts. Thereafter, the linear system of equations can be solved and discrete values of the numerically approximated function (e.g. displacements) are obtained. It must be pointed out, that in IGA, these discrete values belong to control points of the NURBS description and therefore do not necessarily lie on the geometric object [8].

The accuracy of the numerically approximated results depends significantly on the selected refinement strategy. In IGA, methods for increasing the degree of the NURBS basis functions (p-method) and for mesh refinement (h-method) can be applied easily. Furthermore, in IGA, it is possible to increase the degree of the basis functions and simultaneously increase their continuity between elements (k-method). The aforementioned strategies for refinement can be used contemporary (hpk-method) [2; 8]. Further theoretical background and practical hints for the implementation and application of IGA can be found e.g. in [8; 15].

In Figure 2, parts of the numerical analysis applying IGA are shown exemplarily. A straight beam, see Figure 2a) green line, is loaded by a single point load in the mid-span. The deformed configuration, see Figure 2a) black line, is calculated using isogeometric Euler-Bernoulli beam elements. The corresponding NURBS basis functions are shown in Figure 2b).

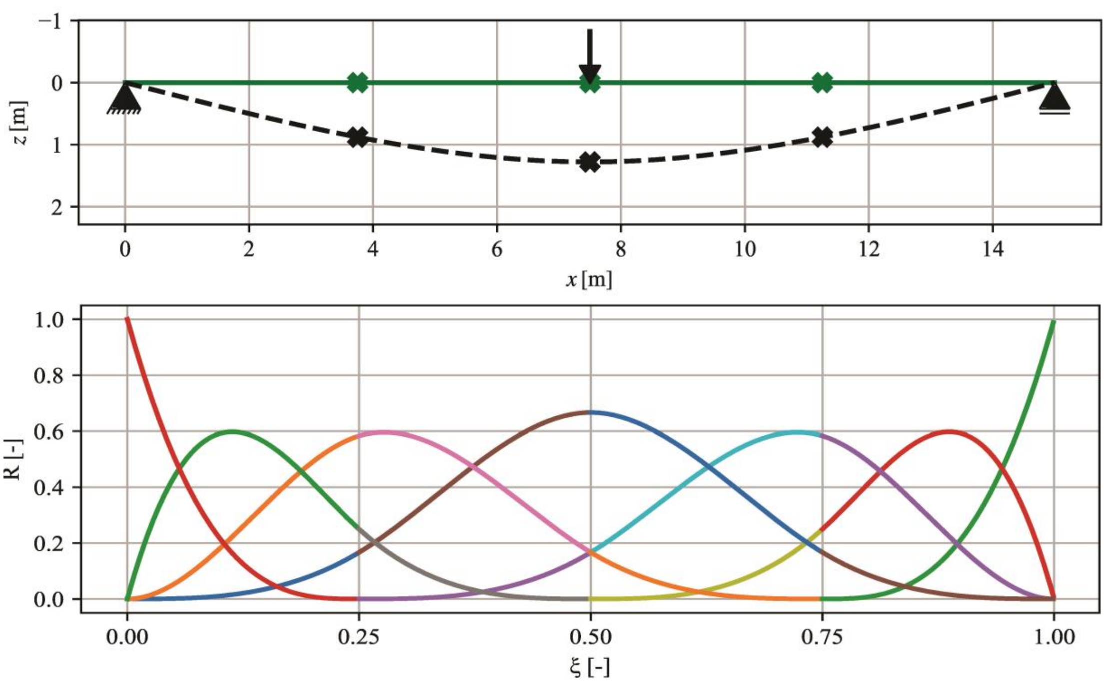

Figure 2: Isogeometric analysis of a straight beam; a) initial (green) and deformed (black) configuration, point load; b) NURBS basis function.

4 Potential advantages of IGA in bridge design and calculation

4.1 CAD-integrated, parametric design and analysis

NURBS-based IGA allows for the calculation of deformations, internal forces, and stresses of a structure, directly using a predefined geometric model. A discretisation and simplification of the geometric model and consequently the generation of an additional numerical model is not necessary [8]. Especially in early planning phases of a bridge, when changes in geometry are performed regularly, this offers considerable procedural advantages. Using NURBS, even complex systems can be described and analysed without a discretisation of the geometry. Due to the high continuity of NURBS basis functions and extended methods of refinement, the numerical analysis using IGA is performed with a lower computational effort [2; 8]. This allows even demanding nonlinear calculations, e.g. for the analysis of solid composite structures, to be performed within an acceptable time-range. An efficient CAD-integrated design and analysis process also provides a good basis for algorithmic optimisation processes. Furthermore, by exploiting the convex-hull property of NURBS curves, surfaces, and solids, bridge structures can be integrated into the landscape automatically, respecting predefined geometric boundary conditions, see Figure 3 [7].

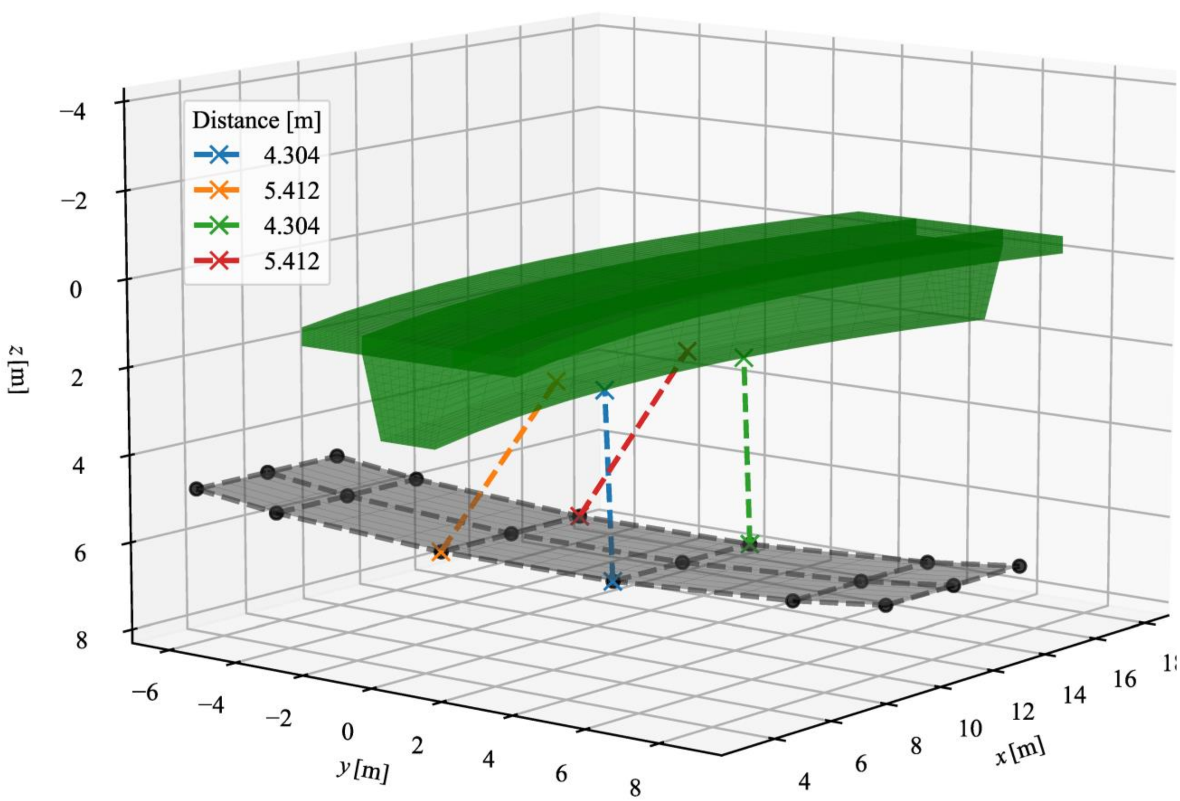

Figure 3: Automatic, numerical calculation of the distance of the lower edge of a bridge superstructure (green) to a road (gray) in the design phase.

The structure of the geometric modelling with NURBS allows for a very simple consideration of composite cross sections or different components in an overall model without the need for complicated and, above all, mesh-dependent coupling descriptions. By embedding sub-geometries in a master geometry, elements with the same or smaller dimensions can be embedded in each other [16]. For example, a beam, surface or solid element can be embedded in a solid element (master element). Comparative calculations with the trimming method have shown that embedding requires more elements, but the results converge towards those of trimming [17].

The application of this theory allows, among other things, the consideration of tendons as cable elements in a bridge superstructure, both for the determination of a realistic total stiffness and for the application of loads at the correct location. The strict application of the embedment theory leads to a geometry generation of the tendons in the parameter space of the master element. In the case of a bridge, for example, the master element represents the solid model of the bridge superstructure. This ensures that the tendons lie within the master structure at every point. It is also possible to check the spacing of the tendons and their compliance with the concrete cover. See Figure 4.

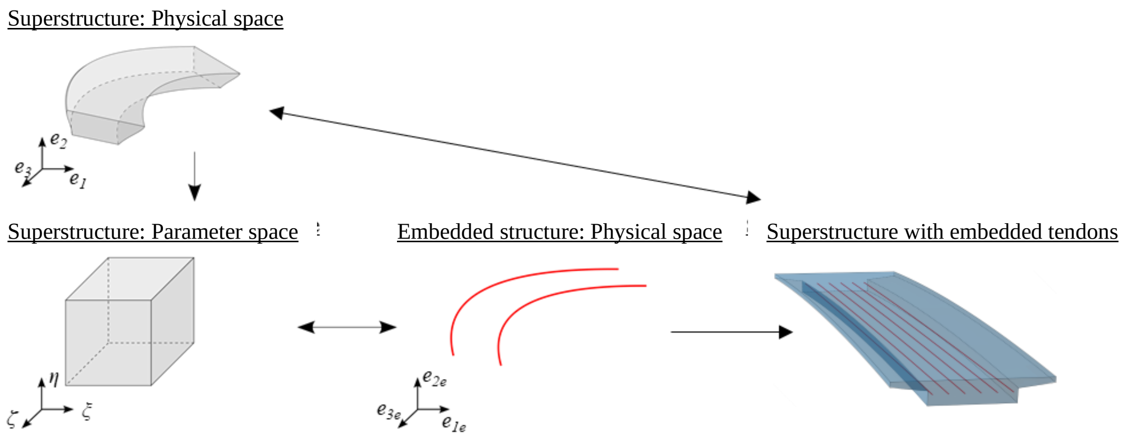

Figure 4: Embedding structures in the parameter space of the superstructure and bridge superstructure with embedded tendons.

The prestressing forces cannot be easily calculated from the embedded cable elements because the input variables, such as the calculation of the length or curvature of the curve, are present in the parameter space of the master element. The curve elements must first be transferred or mapped from the parameter space to the physical space of the master structure, see Figure 5 [16]. This is done using Eq. 9.

$$\mathbf{C}(\bar{\xi}) = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} R_{i,j,k,p,q,r}(\bar{\xi})\,\mathbf{P}_{i,j,k} = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} R_{i,j,k,p,q,r}\left(\sum_{\bar{i}=1}^{\bar{n}} R_{\bar{i},\bar{p}}(\bar{\xi})\,\bar{\mathbf{P}}_{\bar{i}}\right)\mathbf{P}_{i,j,k} \tag{9}$$

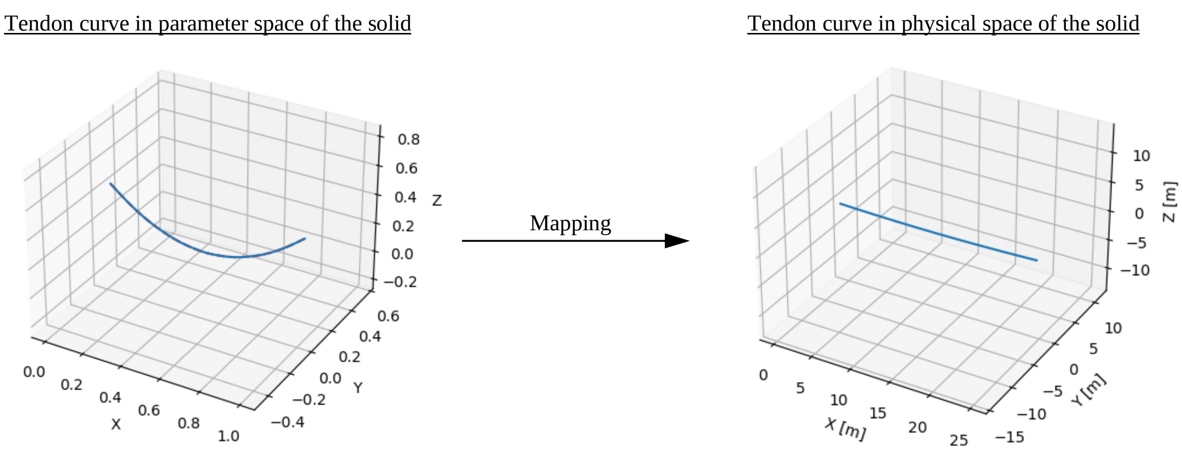

Figure 5: Mapping between parameter space and physical space.

4.2 Geometric reduction

For the mathematical description of a bridge's geometry, free-form modelling methods using NURBS can be applied, as shown in Section 2. These offer the designer the possibility to design reinforced concrete, prestressed concrete and composite components geometrically freely, to dimension them according to the component loads and to use construction materials in a resource-efficient way [18-20]. However, these advantages are currently in opposition to an increased effort for the calculation and dimensioning of these components. The necessary creation of structural analysis models as beam, bar or shell structures must be carried out manually for each design step or after each geometric adjustment of the component.

The authors are currently working on a method which should allow the automated derivation of analysis models from NURBS solid models of (reinforced) concrete members, see [11; 21]. Due to the tensor-product structure of NURBS solids and surfaces and the possibility to easily calculate local derivatives and gradients, it is possible to automatically derive geometrically reduced numerical models from three-dimensional geometric representations. This allows the advantages of a geometrically reduced numerical model to be exploited, for example for the design of reinforced concrete bridge components, while at the same time retaining the digital representation of the three-dimensional model. This procedure is shown in Figure 6 on the example of a bridge superstructure [11; 21].

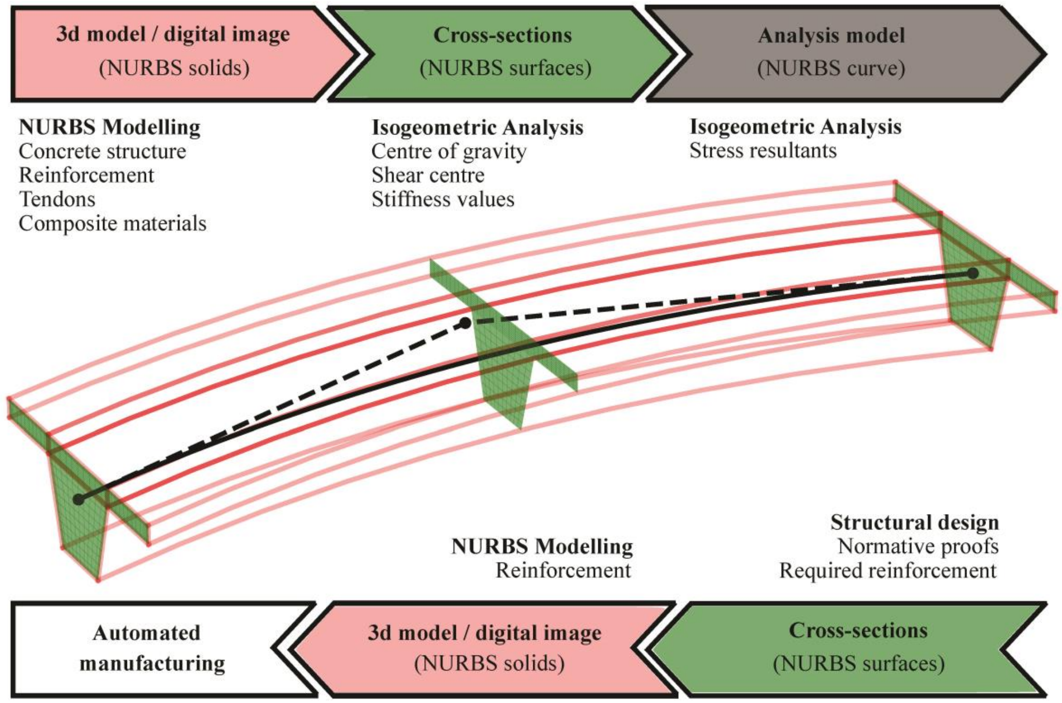

Figure 6: Geometric reduction of a bridge superstructure and consistent data exchange in the framework of IGA [11].

4.3 Consistent data exchange in a digital workflow

When using a single NURBS model for the design and numerical calculation of a bridge superstructure, all digital data of the construction are updated in each design and calculation step and provided for further purposes. They can be used e.g., for the evaluation of a design step in terms of cost, construction material consumption, or CO_2 equivalent. Furthermore, the raw data of the geometric description of structural components may be used for digital fabrication and automated manufacturing. An essential basis for the development of consistently digital and automated processes, see Figure 7, for the production of bridge components is the availability of suitable raw data sets. Depending on the requirements of the sub-processes, these are transferred directly to machines via interfaces or converted into production data in advance. Raw data relevant for a digital manufacturing process is essentially geometric in nature, enriched with semantic information about construction materials, like [22]:

- Position and origin of coordinate systems
- Mathematical description of the geometry (curves, surfaces, solids)
- Construction material properties

Since geometric modelling using NURBS is standard in CAD programs, the exchange of raw data is easy. Normally, this is done using the IGES or STEP data exchange formats [7]. In addition, the exchange of NURBS data sets is also possible within the framework of BIM using the IFC format [23].

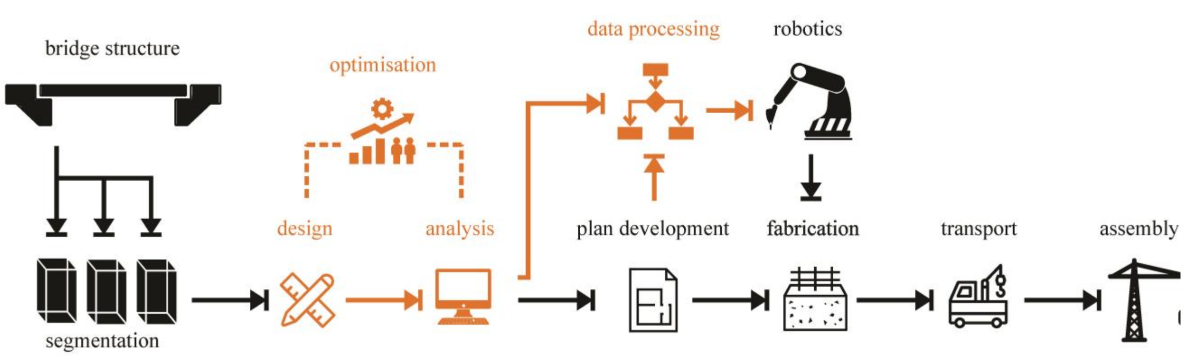

Figure 7: Data exchange and processing in a digital workflow [22].

5 Conclusions and outlook

In this contribution, we summarise the fundamentals of IGA and the state of science and technology for the application of IGA in bridge design. Currently, this CAD-integrated procedure is not used in bridge planning practice and scientific studies only deal with special topics. Nevertheless, geometric modelling and analysis using NURBS and IGA can offer significant advantages in this field. Three of these advantages are presented briefly in this paper. The authors of this paper are currently working on new methods which should enable a practicable application in the practice of bridge construction.

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