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Unannounced Interim Inspections:

Do False Alarms Matter?

-- Proofs and Extensions --

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Abstract

Unannounced Interim Inspections (UIIs) in nuclear plants of the European Union have recently attracted major attention by the International Atomic Energy Agency (IAEA) and by European Atomic Energy Community (EURATOM) in the context of the IAEA/EURATOM Partnership Approach. Therefore, a research project had been organized by the Joint Research Centre in Ispra in collaboration with the Universität der Bundeswehr München in the framework of which the assumptions have been classified which are necessary for a quantitative analysis and a few variants have been studied in detail.
In that project only so-called Attribute Sampling Procedures were considered which means that only errors of the second kind (no detection of the illegal activity), but not those of the first kind (false alarms), where taken into account. It was the purpose of the work presented here to investigate the impact of errors of the first kind on UIIs which may occur if so-called Variable Sampling Procedures are used. Two kinds of planning UIIs are considered: In the sequential one both players, the inspector and the plant operator, decide step by step to inspect resp. to start the illegal activity – if at all. In the hybrid-sequential one the inspector decides at the beginning of the reference time interval where to place his UIIs, whereas the plant operator acts again sequentially.
For two UIIs during the reference time interval equilibria are determined, which generalize the results of the above mentioned research project. It turns out that in both cases, the sequential and hybrid-sequential one, the equilibrium strategies of the inspector and the equilibrium payoffs to both players are the same, but not the equilibrium strategies of the plant operator. We try to present a plausible explanation for this surprising result.

Unannounced Interim Inspections: Do False Alarms Matter? – Proofs and Extensions –

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1 Introduction

Unannounced Interim Inspections (UII) in nuclear plants of the European Union have recently attracted major attention by the International Atomic Energy Agency (IAEA) and by European Atomic Energy Community (EURATOM) in the context of the IAEA/EURATOM Partnership Approach of both organisations. Therefore, a research project had been organized by the Joint Research Centre in Ispra in collaboration with the Universität der Bundeswehr München in the framework of which the assumptions have been classified which are necessary for a quantitative analysis and a few variants have been studied in detail. The results of these analyses have been applied to two kinds of nuclear facilities in one State of the European Union, see [AK09] and [ACK09].

One assumption made in [AK09] and [ACK09] is that only so-called Attribute Sampling Procedures were considered which means that only errors of the second kind (no detection of the illegal activity) are taken into account, but not those of the first kind (false alarms) which cannot be avoided if so-called Variable Sampling Procedures are applied by the inspector. It was the purpose of this work to investigate the impact of the possibility of false alarms, when UIIs are performed at the hand of one concrete model considered in the work mentioned above. The limitation to only one model results from the fact that the modelling effort increases significantly, as will be explained and demonstrated subsequently, if false alarms are taken into account.

Formal models for inspections using Variable Sampling Procedures have been analyzed at various occasions. In particular one variant has been considered in detail in [AC05], where

- UIIs are possible at any time during the reference time interval (continuous time model)
- Both the inspection authority and the operator proceed sequentially: The first decides at the beginning only when to perform the first UII and after it has taken place, when to perform

the second one and so on. The operator decides first whether or not to start the illegal activity immediately or after the first inspection and so on. In other words, the inspector decides about the inspection time points and the operator only whether to start the illegal activity immediately or later.

- The objectives of both players are expressed by the detection time: The inspection authority aims at as short a time possible between the start and the detection of the illegal activity – if there is one – whereas the operator aims at getting it as long as possible.

- For any number of UIIs during the reference time interval Nash equilibria, i.e., equilibrium strategies and payoffs for to both players have been determined as functions of the parameters of the model: the payoff parameters and errors first and second kind probabilities. In particular conditions for legal behavior of the operator have been given.

Here a hybrid-sequential model, i.e., where only the operator acts sequentially, is analyzed. This model has been considered already in the project mentioned above, see [AK09] and [ACK09], for Attribute Sampling Procedures. Since it turned out that in this case both models lead to the same result, i.e., the same equilibrium strategies and payoffs, it was of special interest to find out whether or not this holds also for Variable Sampling Procedures. For this purposes only two UIIs in the reference time interval are considered – for only one UII both models are identical – even though, should it be of major interest, the analysis might be generalized to more than two UIIs.

In the following a quantitative hybrid-sequential continuous time model for two UIIs is developed and Nash equilibria of this model are determined. It turns out that the equilibrium strategy of the inspector and equilibrium payoffs to both players are the same both in the hybrid-sequential and the sequential model, but not the equilibrium strategies of the operator. We try to give a plausible explanation for this surprising result.

2 The model

In the following we present a game theoretical model for Unannounced Interim Inspections. We consider a nuclear facility and two UIIs during the reference time interval (e.g., one year). Furthermore, we consider a so-called hybrid-sequential model, i.e., a model in which the inspector fixes the two time points for his UIIs at the beginning of the reference time interval, whereas the plant operator decides at the beginning of the reference time interval whether to start the illegal activity immediately or not, in the latter case after the first inspection again in the same way, and so on. The objective of the operator is to achieve as long a time possible between the start of the illegal activity and its detection, the latest at the end of the reference time interval („playing for time"); the objective of the inspector is to get this time interval as short as possible.

Let us summarize the assumptions we have made so far, and some additional technical ones:

1. There are two players: operator and inspector.

2. The inspector can perform his inspection at any time point within the reference time interval (we ignore the fact that in reality an inspection extends over some finite time interval). The operator *can* start his illegal activity only right after an inspection, and therefore, the illegal

activity can be detected only at the occasion of the next inspection(s) or with certainty at the Physical Inventory Verification (PIV) at the end of the reference time interval.

3. The inspector will commit – depending on measures taken by him – an error of the first kind (false alarm) and of the second kind (no detection of the illegal activity) with probability α resp. β per inspection.

4. The number of interim inspections is known to the operator. Two unannounced interim inspections are permitted in the facility and the reference time interval.

5. The inspector decides at the beginning of the reference time interval when to perform his inspections. The operator decides at the beginning of the reference time interval whether to start his illegal activity immediately or only right after the inspection(s) – if at all.

6. The payoff to the operator resp. the inspector is proportional to the time between the start of the illegal activity and its detection.

7. The game ends either after the final PIV or after that interim inspection at which the illegal activity – if there is one – is detected.

Since the information structure of this game is much more complicated that that of other games considered in [AK09] and [ACK09], we present first a version without the possibility of errors of the first and second kind, and then a version without the possibility of errors of the first kind (*Attribute Sampling*). In both these variants we consider only the *illegal* game, i.e., the game in which the operator will behave illegally with certainty. If we want to include legal behavior, then we have to define payoffs which evaluate advantages and disadvantages of legal and illegal behavior of the operator. This will be done in section 2.3 where we analyze the general case (*Variable Sampling*).

2.1 First Model: $\alpha = \beta = 0$

Both in the first and second model we consider only the *illegal* game, i.e., the game in which the operator will behave illegally with certainty. If we wanted to include legal behavior, then we would have to define payoffs which evaluate advantages and disadvantages of legal and illegal behavior of the inspector and the operator. This will be done in the third model.

In Figure 1 the extensive form, see [Mye97], of our game is represented graphically. As outlined above, we consider a non-cooperative zero-sum game with the detection time as payoff to the operator.

——————— Figure 1 about here ———————

Let us describe this game in words: At the beginning of the reference time interval (t_3) the inspector decides at which time points t_2 and t_1 during the reference time interval to perform his two UIIs. Time is counted backward for formal mathematical reasons.

The operator decides at t_3 whether to behave illegally ($\bar{l}_3$) or not (l_3). In the latter case he decides again at t_2, i.e., after the first inspection, whether to behave illegally ($\bar{l}_2$) or not (l_2). His information set for a given t_2 contains all possible time points t_1: $t_2 < t_1 < t_0$, thus there are infinitely many information sets, one for each t_2. In Annex 2 we consider a time discrete version of this game in order to demonstrate the structure of the information sets with the help of an example with finitely many pure strategies of both players.

As already mentioned, we consider here the *illegal* game, which means that the operator has to start his illegal activity the latest at time point t_1 if he did not so before. If we denote with g_3 the probability that the operator starts his illegal activity at t_3, and with $g_2(t_2)$ the probability that the operator starts his illegal activity at t_2, then the expected payoff to the operator for fixed (t_2, t_1) and fixed $(g_3, g_2(t_2))$ is

$$Op((t_2, t_1); (g_3, g_2(t_2))) \tag{1}$$
$$= g_3 \cdot (t_2 - t_3) + (1 - g_3) \cdot \Big[g_2(t_2) \cdot (t_1 - t_2) + (1 - g_2(t_2)) \cdot (t_0 - t_1) \Big] .$$

Since it will turn out that there exists an Nash equilibrium strategy of the inspector in pure strategies, we need not introduce mixed strategies of the inspector. Therefore, equilibrium strategies (t_2^*, t_1^*) and $(g_3^*, g_2^*(t_2))$ and payoffs Op^* and $-Op^*$ are determined with the help of the saddle point criterion, see [vNM47],

$$Op((t_2^*, t_1^*); (g_3, g_2(t_2^*))) \le Op((t_2^*, t_1^*); (g_3^*, g_2^*(t_2^*))) \le Op((t_2, t_1); (g_3^*, g_2^*(t_2))) \tag{2}$$

for all $t_3 < t_2 < t_1 < t_0$ and all $g_3, g_2(t_2) \in [0, 1]$. In the following we define

$$Op^* := Op((t_2^*, t_1^*); (g_3^*, g_2^*(t_2^*))) .$$

Before presenting the equilibrium explicitly, let us clarify the nature of the strategies of the operator: Mathematically speaking a *pure* behavioral strategy at t_2 is a mapping of t_2 into the set $\{\bar{l}_2, l_2\}$. Therefore, a behavioral strategy at t_2 is also such a mapping and thus, depends on t_2.

Lemma 1. *An equilibrium of the game represented graphically in Figure 1 is given by*

$$t_1^* - t_2^* = \frac{1}{2} \cdot (t_0 - t_2^*) , \quad t_2^* - t_3 = \frac{1}{3} \cdot (t_0 - t_3) ,$$
$$g_3^* = \frac{1}{3} , \quad g_2^*(t_2) = \frac{1}{2} \quad \textit{for all} \quad t_2 \in (t_3, t_0)$$

with the optimal expected detection time Op^*

$$Op^* = \frac{1}{3} \cdot (t_0 - t_3) .$$

Proof. With $t_1^* - t_2^* = \frac{1}{3} \cdot (t_0 - t_3)$ we get with (1)

$$\begin{aligned} Op((t_2^*, t_1^*); (g_3, g_2(t_2^*))) &= g_3 \cdot \frac{1}{3} \cdot (t_0 - t_3) + (1 - g_3) \cdot \Big[g_2(t_2) \cdot \frac{1}{3} \cdot (t_0 - t_3) \\ &+ (1 - g_2(t_2)) \cdot \frac{1}{3} \cdot (t_0 - t_3) \Big] \\ &= \frac{1}{3} \cdot (t_0 - t_3) = Op^* \end{aligned}$$

and

$$\begin{aligned} Op((t_2, t_1); (g_3^*, g_2^*(t_2))) &= \frac{1}{3} \cdot (t_2 - t_3) + \frac{2}{3} \cdot \Big[\frac{1}{2} \cdot (t_1 - t_2) + \frac{1}{2} \cdot (t_0 - t_1) \Big] = \frac{1}{3} \cdot (t_0 - t_3) \\ &= Op^* , \end{aligned}$$

i.e., both inequalities in (2) are fulfilled with equality. □

Most importantly, we see that $g_2^*(t_2)$ does not depend on t_2. Also, one might have guessed this solution. For $\alpha = 0, \beta > 0$, however, one might hardly guessed it.

For the purpose of comparison we present in Figure 2 the extensive form of the sequential model, i.e., the case where both the inspector and the operator behave sequentially.

——————— Figure 2 about here ———————

The most important difference between the two cases is that in the latter one there exist *subgames* which permit a recursive treatment of the game (which of course is more important for more than two inspections). Nevertheless, the equilibria of both games are the same, see [AC05].

2.2 Second Model: $\alpha = 0$, $\beta > 0$

Before we turn to the Variable Sampling case, we consider the Attribute Sampling case, i.e., $\alpha = 0$ and $\beta > 0$ for the sake of completeness and for methodological reasons.

The extensive form of this more complicated game is given in Figure 3.

——————— Figure 3 about here ———————

The meaning of $\bar{l}_3$, l_3, $\bar{l}_2$, l_2, g_3 and $g_2(t_2)$ is the same as before; β is the probability of not detecting the illegal activity in the course of an inspection if this illegal activity is started right after the previous inspection (or the PIV). Again we consider only the illegal game.

In Figure 4 the reduced form of the game given in Figure 3 is represented graphically.

——————— Figure 4 about here ———————

We see that this reduced game has the same structure as the game for $\beta = 0$, see Figure 1. The expected payoff to the operator for fixed (t_2, t_1) and fixed $(g_3, g_2(t_2))$ is

$$\begin{aligned} & Op((t_2, t_1); (g_3, g_2(t_2))) \\ & \quad = g_3 \cdot \Big[(1 - \beta) \cdot (t_2 - t_3) + \beta \cdot \Big((1 - \beta) \cdot (t_1 - t_3) + \beta \cdot (t_0 - t_3) \Big) \Big] \\ & \qquad + (1 - g_3) \cdot \Big[g_2(t_2) \cdot \Big((1 - \beta) \cdot (t_1 - t_2) + \beta \cdot (t_0 - t_2) \Big) + (1 - g_2(t_2)) \cdot (t_0 - t_1) \Big] \end{aligned}$$

Since the same arguments – see the paragraph before Lemma 1 – hold here as well, we formulate immediately

Lemma 2. *An equilibrium of the game represented graphically in Figure 3 is given by*

$$t_1^* - t_2^* = \frac{1}{2} \cdot (t_0 - t_2^*)\,, \quad t_2^* - t_3 = \frac{1-\beta}{3 - 2\cdot\beta} \cdot (t_0 - t_3)\,,$$

$$g_3^* = \frac{1}{3 - 2\cdot\beta}\,, \quad g_2^*(t_2) = \frac{1}{2} \quad \textit{for all} \quad t_2 \in (t_3, t_0)$$

with the optimal expected detection time $Op^*(\beta)$

$$Op^*(\beta) = \frac{1}{3 - 2\cdot\beta} \cdot (t_0 - t_3)\,.$$

Proof. Since it goes along the same lines as that of Lemma 1, we need not repeat it here. □

As already mentioned, we hardly could have guessed this solution even though the structure of this game is the same as that for $\beta = 0$.

2.3 Third (General) Model: $\alpha > 0$ and $\beta > 0$

If we consider now Variable Sampling Procedures which includes the possibility of errors of the first and second kind, several new aspects have to be taken into account.

From a practical point of view, we assume that the „game“ continues after an error of the first kind – false alarm – has been committed, of course, causing costs to both players. Therefore, the zero sum assumption has to be given up, and more than that, payoff parameters have to be introduced which evaluate the different outcomes of the game. This however, gives us the possibility to answer a question not posed so far: Under which circumstances will the operator be induced to behave legally?

In Figures 5 and 6 the extensive form of our general inspection game is represented graphically, i.e., a game in which illegal as well as legal behavior is possible.

—————— Figures 5 and 6 about here ——————

$\bar{l}_3, l_3, \bar{l}_2$ and l_2 have the same meaning as in the first model. $\bar{l}_1$ means the start of the illegal activity at time point t_1, l_1 means legal behavior at t_1. $1 - \beta$ is the detection probability, α the false alarm probability.

It should be mentioned that we also assume that a false alarm is not possible in the course of an inspection if prior to that inspection an illegal activity was started. This is not a trivial assumption; depending on the details of the inspection procedure alternative assumptions have to be formulated.

Let $\Delta\, t$ be the time interval between start of the – if at all – illegal activity and its detection, the latest at t_0, i.e., at the end of the reference time interval. Then the payoffs to the operator are

0	for legal behavior of the operator and no false alarms
$-f$	for legal behavior of the operator and false alarms
$d \cdot \Delta\, t - b$	for illegal behavior of the operator

and to the inspector

0	for legal behavior of the operator and no false alarms
$-e$	for legal behavior of the operator and false alarms
$-a \cdot \Delta t$	for illegal behavior of the operator

where $0 < e < a \cdot (t_0 - t_3)$, $0 < f < d \cdot (t_0 - t_3)$ and $0 < b$.

Furthermore, for the longest possible detection time $\Delta t = t_0 - t_3$ we have to postulate

$$d \cdot (t_0 - t_3) - b > 0$$

otherwise the operator would not have any incentive to behave illegally at all.

Since for a given time point t_1 the operator has to decide between $\bar{l}_1$ and l_1 according to

$$d \cdot (t_0 - t_1) - b \lesseqgtr 0$$

for all possible situations, see Figure 5, we introduce the decision variable $g_1(t_1)$ meaning

$$g_1(t_1) = \begin{cases} 1 & \text{for} \quad \bar{l}_1 \\ 0 & \text{for} \quad l_1 \end{cases}$$

and can be then reduce the game tree as shown in Figure 7. From the mathematical point of view g_1 should depend on t_1 and t_2, see Figure 5. Due to our special payoff structure, however, g_1 does not depend on t_2.

—————— Figure 7 about here ——————

Since the decision between $\bar{l}_2$ and l_2 is based on the same payoff alternative in both information sets it is sufficient to introduce the same behavioral strategy $g_2(t_2)$ for both information sets.

Then, for fixed (t_2, t_1) and fixed $(g_3, g_2(t_2))$, the expected payoff to the operator is given by

$$\begin{aligned} & Op((t_2, t_1); (g_3, g_2(t_2), g_1(t_1))) \qquad (3) \\ & = d \cdot \Big\{ g_3 \cdot \Big[(1-\beta) \cdot (t_2 - t_3) + \beta \cdot \Big((1-\beta) \cdot (t_1 - t_3) + \beta \cdot (t_0 - t_3) \Big) - \frac{b}{d} \Big] \\ & \quad + (1 - g_3) \cdot \Big[g_2(t_2) \cdot \Big(-\alpha \cdot \frac{f}{d} + (1-\beta) \cdot (t_1 - t_2) + \beta \cdot (t_0 - t_2) - \frac{b}{d} \Big) \\ & \quad + (1 - g_2(t_2)) \cdot \Big(g_1(t_1) \cdot (-2 \cdot \alpha \cdot \frac{f}{d} + (t_0 - t_1) - \frac{b}{d}) \\ & \quad + (1 - g_1(t_1)) \cdot (-2 \cdot \alpha \cdot \frac{f}{d}) \Big) \Big] \Big\} \end{aligned}$$

and to the Inspector

$$\begin{aligned} & In((t_2, t_1); (g_3, g_2(t_2), g_1(t_1))) \qquad (4) \\ &= -a \cdot \Big\{ g_3 \cdot \Big[(1-\beta) \cdot (t_2 - t_3) + \beta \cdot \Big((1-\beta) \cdot (t_1 - t_3) + \beta \cdot (t_0 - t_3) \Big) \Big] \\ &\quad + (1 - g_3) \cdot \Big[g_2(t_2) \cdot \Big(\alpha \cdot \frac{e}{a} + (1-\beta) \cdot (t_1 - t_2) + \beta \cdot (t_0 - t_2) \Big) \\ &\quad + (1 - g_2(t_2)) \cdot \Big(g_1(t_1) \cdot (2 \cdot \alpha \cdot \frac{e}{a} + (t_0 - t_1)) \\ &\quad + (1 - g_1(t_1)) \cdot (2 \cdot \alpha \cdot \frac{e}{a}) \Big) \Big] \Big\} . \end{aligned}$$

Equilibrium strategies and the corresponding payoffs of this non-cooperative two person game are defined by the Nash-conditions, see [Nas51]

$$Op^* = Op((t_2^*, t_1^*); (g_3^*, g_2^*(t_2^*), g_1^*(t_1^*))) \geq Op((t_2^*, t_1^*); (g_3, g_2(t_2^*), g_1(t_1^*))) \qquad (5)$$

$$In^* = In((t_2^*, t_1^*); (g_3^*, g_2^*(t_2^*), g_1^*(t_1^*))) \geq In((t_2, t_1); (g_3^*, g_2^*(t_2), g_1^*(t_1))) \qquad (6)$$

for all $g_3, g_2(t_2), g_1(t_1) \in [0,1]$ and all (t_2, t_1) with $t_3 < t_2 < t_1 < t_0$. Here we assume already, as outlined before, that an equilibrium strategy of the inspector is a pure strategy.

We present a Nash equilibrium of our general game theoretical model in

Theorem 1. *Consider the general game theoretical model developed in the previous section and let the test procedure be unbiased, i.e., $\alpha + \beta < 1$. Then a Nash equilibrium is given as follows*

1. *Under the assumption*

$$\frac{b}{d} \geq \frac{t_0 - t_3}{3 - 2 \cdot \beta} + \frac{f}{d} \cdot \alpha \cdot \frac{3 - \beta}{3 - 2 \cdot \beta} \qquad (7)$$

an equilibrium strategy of the operator is legal behavior, i.e., $g_3^* = g_2^*(t_2) = g_1^*(t_1) = 0$ for all $t_3 < t_2 < t_1 < t_0$, that of the inspector is not unique, but given by the set of all (t_2^*, t_1^*) fulfilling the inequalities

$$\begin{aligned} \frac{1}{1-\beta} \cdot \Big(\frac{b}{d} - 2 \cdot \frac{f}{d} \cdot \alpha - \beta^2 \cdot t_0 \Big) &\geq t_2^* + \beta \cdot t_1^* \\ \frac{b}{d} - \frac{f}{d} \cdot \alpha - \beta \cdot t_0 &\geq (1-\beta) \cdot t_1^* - t_2^* \qquad (8) \\ t_0 - \frac{b}{d} &\leq t_1^* \end{aligned}$$

and the equilibrium payoffs are

$$Op^* = -2 \cdot f \cdot \alpha \quad \text{and} \quad In^* = -2 \cdot e \cdot \alpha .$$

2. *Under the assumptions*

$$\frac{b}{d} < \frac{t_0 - t_3}{3 - 2 \cdot \beta} + \frac{f}{d} \cdot \alpha \cdot \frac{3 - \beta}{3 - 2 \cdot \beta} \qquad \textit{and} \qquad \frac{f}{d} \cdot \frac{\alpha}{1 - \beta} \leq \frac{t_0 - t_3}{3 - 3 \cdot \beta + \beta^2} \tag{9}$$

an equilibrium strategy of the operator is

$$g_3^* = \frac{1}{3 - 2 \cdot \beta}, \quad g_2^*(t_2) = \frac{1}{2} \quad \textit{and} \quad g_1^*(t_1) = 1 \tag{10}$$

for all (t_2, t_1) with $t_3 < t_2 < t_1 < t_0$ and the equilibrium strategy of the inspector is

$$t_1^* - t_2^* = \frac{1 - \beta}{2 - \beta} \cdot (t_0 - t_2^*) - \frac{f}{d} \cdot \frac{\alpha}{2 - \beta} \tag{11}$$

$$t_2^* - t_3 = \frac{1 - \beta}{3 - 2 \cdot \beta} \cdot (t_0 - t_3) - \frac{f}{d} \cdot \alpha \cdot \frac{3 - 3 \cdot \beta + \beta^2}{3 - 2 \cdot \beta} \tag{12}$$

and the equilibrium payoffs are

$$Op^* = d \cdot \frac{1}{3 - 2 \cdot \beta} \cdot (t_0 - t_3) - f \cdot \alpha \cdot \frac{3 \cdot (1 - \beta)}{3 - 2 \cdot \beta} - b \tag{13}$$

$$In^* = -a \cdot \frac{1}{3 - 2 \cdot \beta} \cdot (t_0 - t_3) - \alpha \cdot e \cdot \frac{3 \cdot (1 - \beta)}{3 - 2 \cdot \beta}. \tag{14}$$

The proof of this Theorem is given in Annex 1. □

It should also be mentioned that our Theorem does not cover all possibilities, e.g., the case

$$\frac{f}{d} \cdot \frac{\alpha}{1 - \beta} > \frac{1}{3 - 3 \cdot \beta + \beta^2} \cdot (t_0 - t_3).$$

We will come back to this point in the Discussion.

A remark on dimensions: At first sight it looks as if (7) and (9) depend on the dimension of $t_0 - t_3$. This is not true of course, since d – as a proportionality factor – changes appropriately. From this point of view it would be better to always write $d \cdot (t_0 - t_3)$, but this would lead to more clumsy formulae.

Using the technique of proving the Nash equilibrium for the legal game also for that of the illegal game, one can show immediately that the equilibrium strategy of the inspector for the legal game is also an equilibrium strategy of the inspector for the legal game. In this sense we can consider (11) and (12) as a robust equilibrium strategy.

Let us illustrate this with the help of a numerical example:

$$\frac{b}{d} = \frac{3}{4} = 0.75, \quad 1 - \beta = \beta = \frac{1}{2} = 0.5, \quad \frac{f}{d} \cdot \alpha = 0.1, \quad t_3 = 0, \quad t_1 = 1.$$

Then (7) is fulfilled because of

$$0.75 > \frac{1}{2} + 0.1 \cdot \frac{3}{4} = 0.3 + 0.075 = 0.575.$$

According to (8) the strategy of the inspector in the legal equilibrium is

$$0.6 \geq t_2^* + 0.5 \cdot t_1^* \,, \qquad 0.15 \geq 0.5 \cdot t_1^* - t_2^* \,, \qquad 0.25 \quad \leq \quad t_1^* \,.$$

Furthermore, according to (11) the *illegal strategy* (t_2^*, t_1^*) of the inspector is given by

$$t_2^* = \frac{1}{4} - 0.1 \cdot \frac{3 - 3/2 + 1/4}{2} \quad \text{and} \quad t_1^* - t_2^* = 0.33 \cdot (1 - t_2^*) - \frac{0.1}{1.5} \,,$$

which gives $(t_2^*, t_1^*) = (0.16, 0.37)$. In Figure 8 this case is represented graphically.

——————— Figure 8 about here ———————

We see the rather complicated domain for the legal equilibria – shaded area – and the unique illegal equilibrium in the midst of it. In a similar case M. Kilgour called this area *cone of deterrence* by, see [Kil92].

3 Discussion

Whereas we considered in this paper a hybrid-sequential inspection model as explained in the second section, Avenhaus and Canty, see [AC05], studied a sequential model where also the inspector decides at the beginning of the reference time interval only at which time point t_2 to inspect, and at t_2 at which time point t_1 to inspect the second time. It should just be mentioned that in that paper the general case of $k > 1$ inspections during the reference time interval was analyzed.

Surprisingly enough at least at the first sight, the equilibrium of the sequential game is very close to that obtained here: The equilibrium strategy of the inspector as well as the equilibrium payoffs to both players are the same, whereas the equilibrium strategy of the operator in case of illegal behavior is

$$g_3^* = \frac{1}{3 - 2 \cdot \beta} \,, \quad g_2^*(t_2) = \frac{2 \cdot (1 - \alpha) - \beta}{2 \cdot (1 - \alpha) \cdot (2 - \beta)} \quad \text{for all} \quad t_2 \in (t_3, t_0) \,,$$

see [AC05], in contrast to (10) which is independent of α.

One may explain this surprising result as follows: For the inspector there is only one advantage in the sequential variant as compared to the hybrid-sequential one which exists only if both types of errors are possible: Whereas in both variants without first kind errors (but eventually second kind errors) the inspector does not know after the first inspection without detection of the illegal activity whether or not it took place, after a false alarm and its clarification he does know that there was no illegal activity. In the sequential variant therefore he can use this information for the planning of the second inspection, whereas this is not possible in the hybrid-sequential variant. The operator, on his side, reacts to this difference by an appropriately modified equilibrium strategy such that the advantage of the inspector is neutralized.

A weak point of this argument is that without both error types we also have the situation that after inspection the inspector knows whether or not an illegal activity took place, but in both variants the equilibrium strategies of both players are the same. Maybe these games are too simple to contain as subtle differences as described above.

Two additional remarks: First, in our Theorem we did not consider the case

$$\frac{b}{d} < \frac{t_0 - t_3}{3 - 2 \cdot \beta} + \frac{f}{d} \cdot \alpha \cdot \frac{3 - \beta}{3 - 2 \cdot \beta} \qquad \text{and} \qquad \frac{f}{d} \cdot \frac{\alpha}{1 - \beta} > \frac{t_0 - t_3}{3 - 3 \cdot \beta + \beta^2} \, .$$

For the sequential variant this case was considered in [AC05]. There, it led to the equilibrium strategy $t_2^* = 0$ of the inspector which is practically not feasible. Let us mention that the case $t_2 = 0$ is excluded in our model, since we assumed a priori $t_3 < t_2 < t_1 < t_0$. However, we assume that the same would happen here. A game theoretical analysis of this case would require the introduction of mixed strategies for the inspector. We think that this effort is not justified in this unrealistic case.

Second, since our topic is the impact of errors of the first kind on UIIs, let us conclude with two observations derived from our results.

- Even though we have to model the inspection problem as a non-zero-sum game, the equilibrium payoffs (13) and (14) demonstrate that only the additional terms containing α are non-zero sum, whereas the other terms are essential zero-sum (w.l.o.g. take $a = d = 1$).
- Whereas the equilibrium strategy of the operator does not depend on α, that of the inspector does. It enters the equilibrium points of time t_2^* and t_1^* for inspections in the order $\alpha \cdot \frac{f}{d}$, which is supposed to be very small compared to the other terms.

Therefore, we may conclude that even though errors of the first kind may occur, and the subsequent false alarms have to be clarified, for *planning purposes* they may be ignored.

4 Acknowledgement

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5 Annex 1: Proof of Theorem 1

5.1 Proof for the legal game

Let us start with the legal case. i.e., 1. Since the payoff to the operator in case of legal behavior is $-2 \cdot \alpha \cdot f$, he will choose this strategy in equilibrium only if with (13)

$$-2 \cdot \alpha \cdot f > d \cdot \frac{1}{3 - 2 \cdot \beta} \cdot (t_0 - t_3) - f \cdot \alpha \cdot \frac{3 \cdot (1-\beta)}{3 - 2 \cdot \beta} - b$$

which leads immediately to (7). Whereas the equilibrium condition (6) for the inspector is fulfilled as equality, the proof of (8) is more complicated. Let us first write down the equilibrium condition (5) for the legal case:

$$\begin{aligned} -2 \cdot \alpha \cdot f \geq \;& d \cdot \Big\{ g_3 \cdot \Big[(1-\beta) \cdot (t_2^* - t_3) + \beta \cdot \Big((1-\beta) \cdot (t_1^* - t_3) + \beta \cdot (t_0 - t_3) \Big) - \frac{b}{d} \Big] \\ & + (1 - g_3) \cdot \Big[g_2(t_2^*) \cdot \Big(-\alpha \cdot \frac{f}{d} + (1-\beta) \cdot (t_1^* - t_2^*) + \beta \cdot (t_0 - t_2^*) - \frac{b}{d} \Big) \\ & + (1 - g_2(t_2^*)) \cdot \Big(g_1(t_1^*) \cdot (-2 \cdot \alpha \cdot \frac{f}{d} + (t_0 - t_1^*) - \frac{b}{d}) \\ & + (1 - g_1(t_1^*)) \cdot (-2 \cdot \alpha \cdot \frac{f}{d}) \Big) \Big] \Big\} \end{aligned} \tag{15}$$

for all $g_3, g_2(t_2), g_1(t_1) \in [0,1]$. We show now that this condition is equivalent to

$$-2 \cdot \alpha \cdot f \;\geq\; d \cdot \Big[(1-\beta) \cdot (t_2^* - t_3) + \beta \cdot \Big((1-\beta) \cdot (t_1^* - t_3) + \beta \cdot (t_0 - t_3) \Big) \Big] - b \tag{16}$$

$$-2 \cdot \alpha \cdot f \;\geq\; d \cdot \Big[-\alpha \cdot \frac{f}{d} + (1-\beta) \cdot (t_1^* - t_2^*) + \beta \cdot (t_0 - t_2^*) \Big] - b \tag{17}$$

$$-2 \cdot \alpha \cdot f \;\geq\; d \cdot \Big[-2 \cdot \alpha \cdot \frac{f}{d} + (t_0 - t_1^*) \Big] - b \tag{18}$$

$$-2 \cdot \alpha \cdot f \;\geq\; d \cdot \Big[-2 \cdot \alpha \cdot \frac{f}{d} \Big] \tag{19}$$

$\Longrightarrow$: Since (15) holds for all $g_3, g_2(t_2), g_1(t_1) \in [0,1]$, we get (16) – (19) by successively choosing $g_3 = 1$, $g_3 = 0$ and $g_2(t_2) = 1$, and $g_3 = g_2(t_2) = 0$ and $g_1(t_1) = 1$ and finally $g_3 = g_2(t_2) = g_1(t_1) = 0$.

$\Longleftarrow$: Let us multiply (19) by $1-g_1(t_1^*)$ and (18) by $g_1(t_1^*)$ and add the two inequalities. This gives

$$-2\cdot\alpha\cdot f \quad\geq\quad d\cdot\Big(g_1(t_1^*)\cdot(-2\cdot\alpha\cdot\frac{f}{d}+(t_0-t_1^*)-\frac{b}{d})+\,(1-g_1(t_1^*))\cdot(-2\cdot\alpha\cdot\frac{f}{d})\Big)\,.$$

Now let us multiply this inequality by $1-g_2(t_2^*)$ and (17) by $g_2(t_2^*)$ and add these two inequalities. This gives

$$\begin{aligned}-2\cdot\alpha\cdot f \quad\geq\quad & d\cdot\Big[g_2(t_2^*)\cdot\Big(-\alpha\cdot\frac{f}{d}+(1-\beta)\cdot(t_1^*-t_2^*)+\beta\cdot(t_0-t_2^*)-\frac{b}{d}\Big)\\ &+\quad(1-g_2(t_2^*))\cdot\Big(g_1(t_1^*)\cdot(-2\cdot\alpha\cdot\frac{f}{d}+(t_0-t_1^*)-\frac{b}{d})\\ &\qquad+(1-g_1(t_1^*))\cdot(-2\cdot\alpha\cdot\frac{f}{d})\Big)\Big]\,.\end{aligned}$$

Finally let us multiply this inequality by g_3 and (16) by $(1-g_3)$ and add these two inequalities. This gives (15). Thus the equivalence is shown.

Since relation (19) is always fulfilled as an equality, there remain the three inequalities (16) – (18), which are equivalent to (8). □

5.2 Proof for the illegal game

Let us now consider the illegal game , i.e., 2. We have to show that the inequalities (5) and (6) hold. Let us start with (6). We have with (4) and (10)

$$\begin{aligned}&In((t_2,t_1);(g_3^*,g_2^*(t_2),g_1^*(t_1)))\\ =\quad&-a\cdot\Big\{\frac{1}{3-2\cdot\beta}\cdot\Big[(1-\beta)\cdot(t_2-t_3)+\beta\cdot\Big(\cdot(1-\beta)\cdot(t_1-t_3)+\beta\cdot(t_0-t_3)\Big)\Big]\\ &+\frac{2\cdot(1-\beta)}{3-2\cdot\beta}\cdot\Big[\frac{1}{2}\cdot\Big(\alpha\cdot\frac{e}{a}+(1-\beta)\cdot(t_1-t_2)+\beta\cdot(t_0-t_2)\Big)\\ &\qquad+\frac{1}{2}\cdot\Big(2\cdot\alpha\cdot\frac{e}{a}+(t_0-t_1)\Big)\Big]\Big\}\\ =\quad&\frac{-a}{3-2\cdot\beta}\cdot\Big\{(1-\beta)\cdot(t_2-t_3)+\beta\cdot\Big((1-\beta)\cdot(t_1-t_3)+\beta\cdot(t_0-t_3)\Big)\\ &\quad+(1-\beta)^2\cdot(t_1-t_2)+(1-\beta)\cdot\beta\cdot(t_0-t_2)+(1-\beta)\cdot(t_0-t_1)+3\cdot\alpha\cdot\frac{e}{a}\cdot(1-\beta)\Big\}\,.\end{aligned}$$

Collecting the terms with t_i gives

$$\begin{aligned}&In((t_2,t_1);(g_3^*,g_2^*(t_2),g_1^*(t_1)))\\ =\quad&\frac{-a}{3-2\cdot\beta}\cdot\Big\{t_3\cdot[-(1-\beta)-\beta\cdot(1-\beta)-\beta^2]+t_2\cdot[1-\beta-(1-\beta)^2-\beta\cdot(1-\beta)]\\ +\quad&t_1\cdot[\beta\cdot(1-\beta)+(1-\beta)^2-(1-\beta)]\\ +\quad&t_0\cdot[\beta^2+\beta\cdot(1-\beta)+(1-\beta)]+3\cdot\alpha\cdot\frac{e}{a}\cdot(1-\beta)\Big\}\end{aligned}$$

and finally

$$In((t_2,t_1);(g_3^*,g_2^*(t_2),g_1^*(t_1))) = -a\cdot\frac{1}{3-2\cdot\beta}\cdot(t_0-t_3)-\alpha\cdot e\cdot\frac{3\cdot(1-\beta)}{3-2\cdot\beta}=In^*\,,$$

i.e., (6) is fulfilled as equality for all $t_3 < t_2 < t_1 < t_0$.

Let us consider (5). With (11) and (12) we get

$$\begin{aligned}
t_1^*-t_2^* &= \frac{1-\beta}{2-\beta}\cdot(t_0-t_3+t_3-t_2^*)-\frac{f}{d}\cdot\frac{\alpha}{2-\beta}\\
&= \frac{1-\beta}{2-\beta}\cdot(t_0-t_3)+\frac{1-\beta}{2-\beta}\cdot\Big(-\frac{1-\beta}{3-2\cdot\beta}\cdot(t_0-t_3)+\frac{f}{d}\cdot\alpha\cdot\frac{3-3\cdot\beta+\beta^2}{3-2\cdot\beta}\Big)\\
&- \frac{f}{d}\cdot\frac{\alpha}{2-\beta}\\
&= \frac{1-\beta}{3-2\cdot\beta}\cdot(t_0-t_3)-\frac{f}{d}\cdot\alpha\cdot\frac{1}{2-\beta}\cdot\Big(1-\frac{(1-\beta)\cdot(3-3\cdot\beta+\beta^2)}{3-2\cdot\beta}\Big) \qquad (20)
\end{aligned}$$

and therewith

$$t_1^*-t_3=t_1^*-t_2^*+t_2^*-t_3=2\cdot\frac{1-\beta}{3-2\cdot\beta}\cdot(t_0-t_3)-\frac{f}{d}\cdot\alpha\cdot\frac{3-\beta}{3-2\cdot\beta} \qquad (21)$$

and therewith

$$t_0-t_1^*=t_0-t_3+t_3-t_1^*=\frac{1}{3-2\cdot\beta}\cdot(t_0-t_3)+\frac{f}{d}\cdot\alpha\cdot\frac{3-\beta}{3-2\cdot\beta}\,. \qquad (22)$$

Therefore, the factor of $g_1(t_1^*)$ in $Op((t_2^*,t_1^*);(g_3,g_2(t_2^*),g_1(t_1^*)))$ according to (3) is

$$t_0-t_1^*-\frac{b}{d}=\frac{1}{3-2\cdot\beta}\cdot(t_0-t_3)+\frac{f}{d}\cdot\alpha\cdot\frac{3-\beta}{3-2\cdot\beta}-\frac{b}{d}>0$$

because of the left hand inequality of (9). Thus, the right hand side of (5) is maximized by $g_1^*(t_1^*)=1$.

Furthermore, for the factor of $g_2(t_2^*)$ in (3) we have

$$\begin{aligned}
& -\alpha\cdot\frac{f}{d}+(1-\beta)\cdot(t_1^*-t_2^*)+\beta\cdot(t_0-t_2^*)+2\cdot\alpha\cdot\frac{f}{d}-(t_0-t_1^*)\\
= \quad & \alpha\cdot\frac{f}{d}+t_1^*-t_2^*-(1-\beta)\cdot(t_0-t_1^*)\\
\overset{(20),(22)}{=} \quad & \alpha\cdot\frac{f}{d}+\frac{1-\beta}{3-2\cdot\beta}\cdot(t_0-t_3)-\frac{f}{d}\cdot\alpha\cdot\frac{1}{2-\beta}\cdot\Big(1-\frac{(1-\beta)\cdot(3-3\cdot\beta+\beta^2)}{3-2\cdot\beta}\Big)\\
- \quad & (1-\beta)\cdot\Big[\frac{1}{3-2\cdot\beta}\cdot(t_0-t_3)+\frac{f}{d}\cdot\alpha\cdot\frac{3-\beta}{3-2\cdot\beta}\Big]\\
= \quad & 0\,,
\end{aligned}$$

and consequently, for the factor of g_3 in (3), taking into account the results just obtained,

$$(1-\beta)\cdot(t_2^*-t_3)+\beta\cdot(1-\beta)\cdot(t_1^*-t_3)+\beta^2\cdot(t_0-t_3)+2\cdot\frac{f}{d}\cdot\alpha-(t_0-t_1^*)$$

$$\overset{(12),(21),(22)}{=} \quad 2\cdot\frac{f}{d}\cdot\alpha+(1-\beta)\cdot\Big[\frac{1-\beta}{3-2\cdot\beta}\cdot(t_0-t_3)-\frac{f}{d}\cdot\alpha\cdot\frac{3-3\cdot\beta+\beta^2}{3-2\cdot\beta}\Big]$$

$$+ \quad \beta\cdot(1-\beta)\cdot\Big[2\cdot\frac{1-\beta}{3-2\cdot\beta}\cdot(t_0-t_3)-\frac{f}{d}\cdot\alpha\cdot\frac{3-\beta}{3-2\cdot\beta}\Big]+\beta^2\cdot(t_0-t_3)$$

$$- \quad \frac{1}{3-2\cdot\beta}\cdot(t_0-t_3)-\frac{f}{d}\cdot\alpha\cdot\frac{3-\beta}{3-2\cdot\beta}$$

which gives after some lengthy calculations zero as well.

Therefore, we get

$$\begin{aligned}
&Op((t_2^*,t_1^*);(g_3,g_2(t_2^*),g_1(t_1^*)))\\
&\qquad = d\cdot\left(-2\cdot\frac{f}{d}\cdot\alpha+t_0-t_1^*\right)-b\\
&\qquad = d\cdot\left(-2\cdot\frac{f}{d}\cdot\alpha+\frac{1}{3-2\cdot\beta}\cdot(t_0-t_3)+\frac{f}{d}\cdot\alpha\cdot\frac{3-\beta}{3-2\cdot\beta}\right)-b\\
&\qquad = Op^*\,,
\end{aligned}$$

for all $g_3, g_2(t_2^*), g_1(t_1^*) \in [0,1]$, i.e., (5) is fulfilled as equation.

We still have to show $t_3 < t_2^* < t_1^* < t_0$. First, $t_3 < t_2^*$ is with (12) equivalent to

$$(1-\beta)\cdot(t_0-t_3)-\frac{f}{d}\cdot\alpha\cdot(3-3\cdot\beta+\beta^2)>0$$

which is equivalent to the right hand inequality (9). Second, $t_2^* < t_1^*$ is with (20) equivalent to

$$(1-\beta)\cdot(t_0-t_3)-\frac{f}{d}\cdot\alpha\cdot\beta\cdot(2-\beta)>0\,. \tag{23}$$

Because of

$$3-3\cdot\beta+\beta^2-\beta\cdot(2-\beta)=3-5\cdot\beta+2\cdot\beta^2=(3-2\cdot\beta)\cdot(1-\beta)>0$$

we obtain

$$3-3\cdot\beta+\beta^2>\beta\cdot(2-\beta)$$

and with the right hand inequality of (9)

$$\frac{f}{d}\cdot\frac{\alpha}{1-\beta}\le\frac{t_0-t_3}{3-3\cdot\beta+\beta^2}<\frac{t_0-t_3}{\beta\cdot(2-\beta)}\,,$$

which is equivalent to (23). Finally, $t_1^* < t_0$ is with (22) equivalent to

$$t_0-t_3+\frac{f}{d}\cdot\alpha\cdot(3-\beta)>0$$

which is fulfilled anyhow. □

6 Annex 2: Time Discrete Version of the First Model

In this Annex we consider a time discrete version of the First Model in order to demonstrate the structure of the information sets with the helps of an example with finitely many pure strategies of both players.

The model is taken from [AK09], but even more simplified: There are only four possible time points for inspection, namely 1,2,3 and 4. Note that here time is counted forwardly, consistent with [AK09]. An illegal activity of the operator can be started at 0,1,2,3 or 4. Again, we consider only the *illegal* game. We consider two inspections the time points of which are fixed by the inspector at the beginning of the reference time interval. Thus, his pure strategies are (1,2), (1,3), (1,4), (2,3), (2,4) and (3,4). The extensive form of this game is represented graphically in Figure 9.

—————— Figure 9 about here ——————

We see that the operator has three non-trivial information sets. We denote the behavioral strategies at these information sets by $(g_0, 1-g_0)$, $(g_1, 1-g_1)$ and $(g_2, 1-g_2)$. The mixed strategy of the inspector is denoted by

$$\mathbf{p} := (p_{(1,2)}, p_{(1,3)}, p_{(1,4)}, p_{(2,3)}, p_{(2,4)}, p_{(3,4)})^T \qquad \text{with} \qquad \sum_{(i,j): i=1,2,3,\, i<j\leq 4} p_{(i,j)} = 1\,.$$

The payoff to the operator at the end nodes, i.e., the time between the start of the illegal activity and its detection, are given in Figure 9.

Thus, the expected payoff $Op(\mathbf{q},\mathbf{p})$ to the operator is given by

$$\begin{aligned} Op(\mathbf{q},\mathbf{p}) &= g_0 \cdot [\, 1 \cdot (p_{(1,2)} + p_{(1,3)} + p_{(1,4)}) + 2 \cdot (p_{(2,3)} + p_{(2,4)}) + 3 \cdot p_{(3,4)}\,] \\ &+ (1-g_0) \cdot \Big[p_{(1,2)} \cdot (1 \cdot g_1 + 3 \cdot (1-g_1)) + p_{(1,3)} \cdot (2 \cdot g_2 + 2 \cdot (1-g_2)) \\ &\qquad + p_{(1,4)} \cdot (3 \cdot g_1 + 1 \cdot (1-g_1)) \\ &\qquad + p_{(2,3)} \cdot (1 \cdot g_2 + 2 \cdot (1-g_2)) \\ &\qquad + p_{(2,4)} \cdot (2 \cdot g_2 + 1 \cdot (1-g_2)) + 1 \cdot p_{(3,4)} \Big]\,. \end{aligned} \tag{24}$$

An equilibrium of this game is given by the following

Lemma 3. *Given the non-cooperative zero-sum game represented graphically in Figure 9. An equilibrium is given by*

$$\begin{aligned} p^*_{(1,2)} &= p^*_{(1,4)} = \frac{1}{6}\,, \quad p^*_{(1,3)} = 0\,, \quad p^*_{(2,3)} = p^*_{(2,4)} = \frac{1}{3}\,, \quad p^*_{(3,4)} = 0\,, \\ g^*_0 &= \frac{1}{3}\,, \quad g^*_1 = g^*_2 = \frac{1}{2}\,, \\ Op^* &= Op(\mathbf{q}^*,\mathbf{p}^*) = \frac{5}{3}\,. \end{aligned}$$

Proof. We have to show that the saddle point criterion

$$Op(\mathbf{q},\mathbf{p}^*) \le Op(\mathbf{q}^*,\mathbf{p}^*) \le Op(\mathbf{q}^*,\mathbf{p}) \tag{25}$$

is fulfilled for all $\mathbf{q} := (g_0, g_1, g_2) \in [0,1]^3$ and all $\mathbf{p}$. For the right hand side of (25) we get with (24)

$$\begin{aligned} Op(\mathbf{q}^*,\mathbf{p}) &= \frac{1}{3}\cdot[\,p_{(1,2)} + p_{(1,3)} + p_{(1,4)} + 2\cdot(p_{(2,3)} + p_{(2,4)}) + 3\cdot p_{(3,4)}\,] \\ &+ \frac{2}{3}\cdot\Big[\,2\cdot p_{(1,2)} + 2\cdot p_{(1,3)} + 2\cdot p_{(1,4)} + \frac{3}{2}\cdot p_{(2,3)} + \frac{3}{2}\cdot p_{(2,4)} + p_{(3,4)}\,\Big] = \frac{5}{3} \end{aligned}$$

for all $\mathbf{p}$.

For the left hand side of (25) we get with (24)

$$\begin{aligned} Op(\mathbf{q},\mathbf{p}^*) &= g_0\cdot\frac{10}{6} + (1-g_0)\cdot\Big[\frac{1}{6}\cdot(3-2\cdot g_1) + \frac{1}{6}\cdot(1+2\cdot g_1) \\ &\qquad + \frac{1}{3}\cdot(2-g_2) + \frac{1}{3}\cdot(1+g_2)\Big] \\ &= \frac{10}{6} = \frac{5}{3} \end{aligned}$$

for all $\mathbf{q}$, which completes the proof. □

We see that in equilibrium the behavioral strategies at the information sets I and II is the same $g_1^* = g_2^* = \frac{1}{2}$. This corresponds to the result $g_2^*(t_2) = g_2^*$ for all t_2 in the time continuous game, see Lemma 1.

Let us mention that the normal form of this game is more complicated than assumed at first sight: The strategies of the operator are *not* simply $\bar{l}_0, l_0\,\bar{l}_1$ and $l_0\,l_1\,\bar{l}_2$, since in case the operator decides to behave illegally after the first inspection he has different options according to whether the first inspection took place at 1 at 2 or at 3. Because of the three non-trivial information sets we have eight pure strategies of the operator, five of which have to be taken into account explicitly.

Finally, let us compare the time-discrete hybrid-sequential game as given by Figure 9 with the corresponding sequential game the extensive form of which is given by Figure 10.

——————— Figure 10 about here ———————

With the behavioral strategies of the operator and the mixed strategy of the inspector as given in the Figure, the expected payoff to the operator, i.e., the time between the start of the illegal activity and its detection, is given by

$$\begin{aligned} & p_1\cdot\Big[1\cdot g_0 + (1-g_0)\cdot[\,q_1\cdot(1\cdot g_1 + 3\cdot(1-g_1)) + 2\cdot q_2 + q_3\cdot(3\cdot g_1 + 1\cdot(1-g_1))\Big] \\ + \;& p_2\cdot\Big[2\cdot g_0 + (1-g_0)\cdot[\,r_1\cdot(1\cdot g_1 + 2\cdot(1-g_1)) + r_2\cdot(2\cdot g_1 + 1\cdot(1-g_1))\Big] \\ + \;& p_3\cdot\Big[3\cdot g_0 + 1\cdot(1-g_0)\Big]. \end{aligned}$$

Let us compare this payoff with that of the hybrid-sequential game. We identify

$$\begin{aligned}
p_1 \cdot q_1 &= p_{(1,2)}\,, \quad p_1 \cdot q_2 = p_{(1,3)}\,, \quad p_1 \cdot q_3 = p_{(1,4)}\,, \\
p_2 \cdot r_1 &= p_{(2,3)}\,, \quad p_2 \cdot r_2 = p_{(2,4)}\,, \\
p_3 &= p_{(3,4)}\,.
\end{aligned}$$

This means also because of the normalizations

$$p_{(1,2)} + p_{(1,3)} + p_{(1,4)} = p_1 \qquad \text{and} \qquad p_{(2,3)} + p_{(2,4)} = p_2\,,$$

thus, we have a one-to-one correspondence between the mixed extensions of both variants (5 independent variables each). Since the behavioral strategies of the operator are anyhow the same in both cases, we also get the same Nash equilibria.

7 Annex 3: Figures

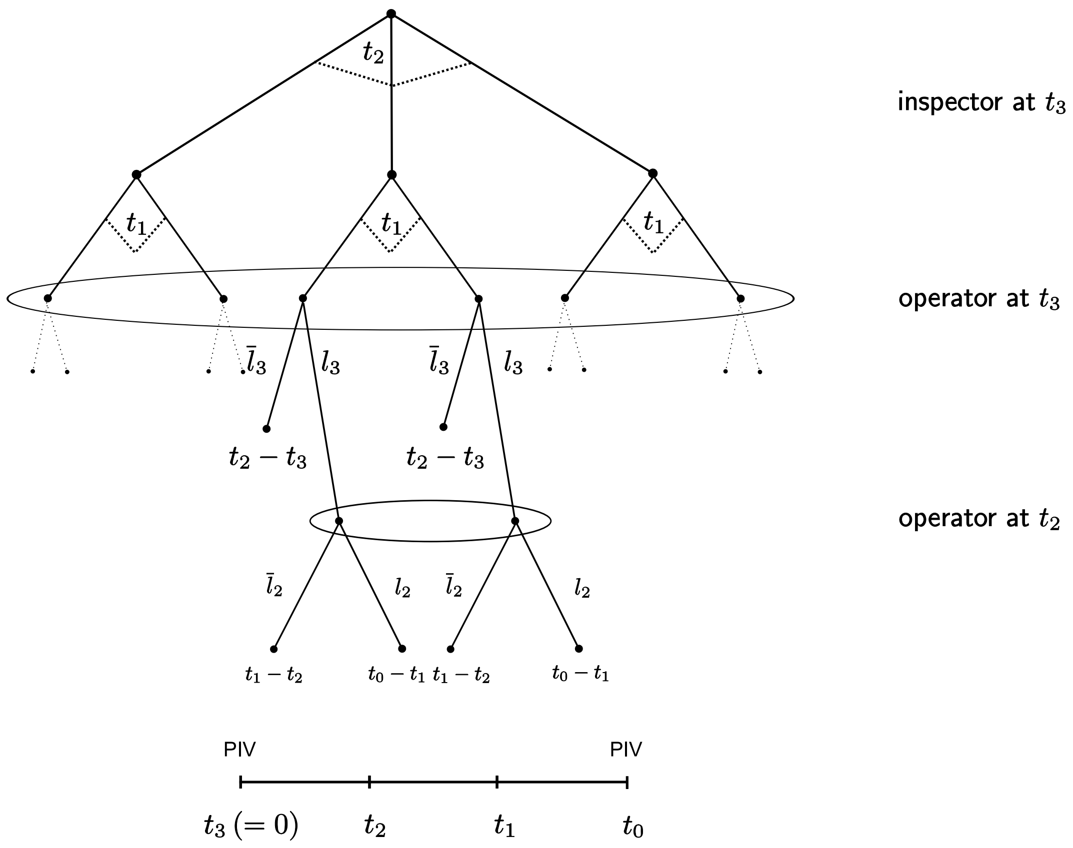


Figure 1: Graphical representation of the extensive form of the *hybrid-sequential illegal* inspection game with two inspections during the reference time interval and without errors of the first and second kind, i.e., $\alpha = \beta = 0$.

Furthermore,

- $\bar{l}_3$: start of the illegal activity at time point t_3. l_3: delay of the illegal activity.
- $\bar{l}_2$: start of the illegal activity at time point t_2. l_2: delay of the illegal activity.
- g_3: the probability to start the illegal activity at time point t_3. $g_2(t_2)$: the probability to start the illegal activity at time point t_2, if it was not yet started at t_3.
- The payoffs to the operator, i.e., the detection times, are depicted at the end nodes.

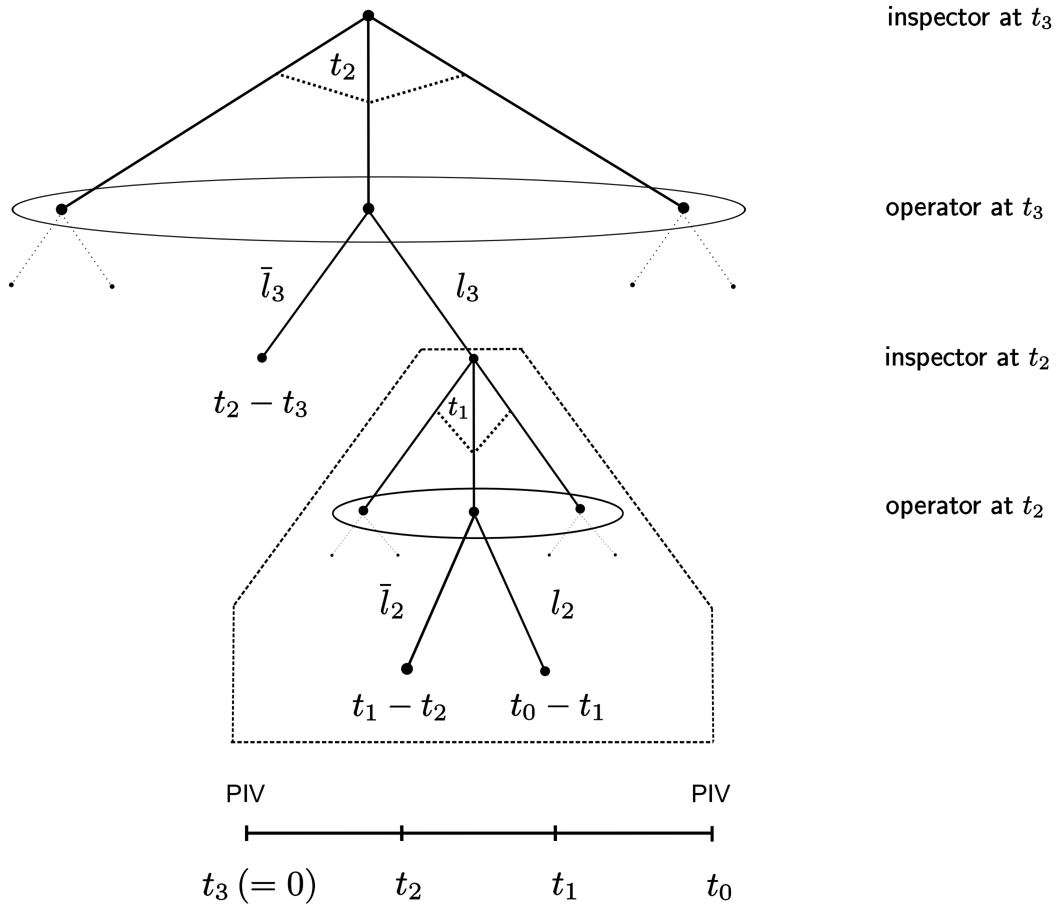
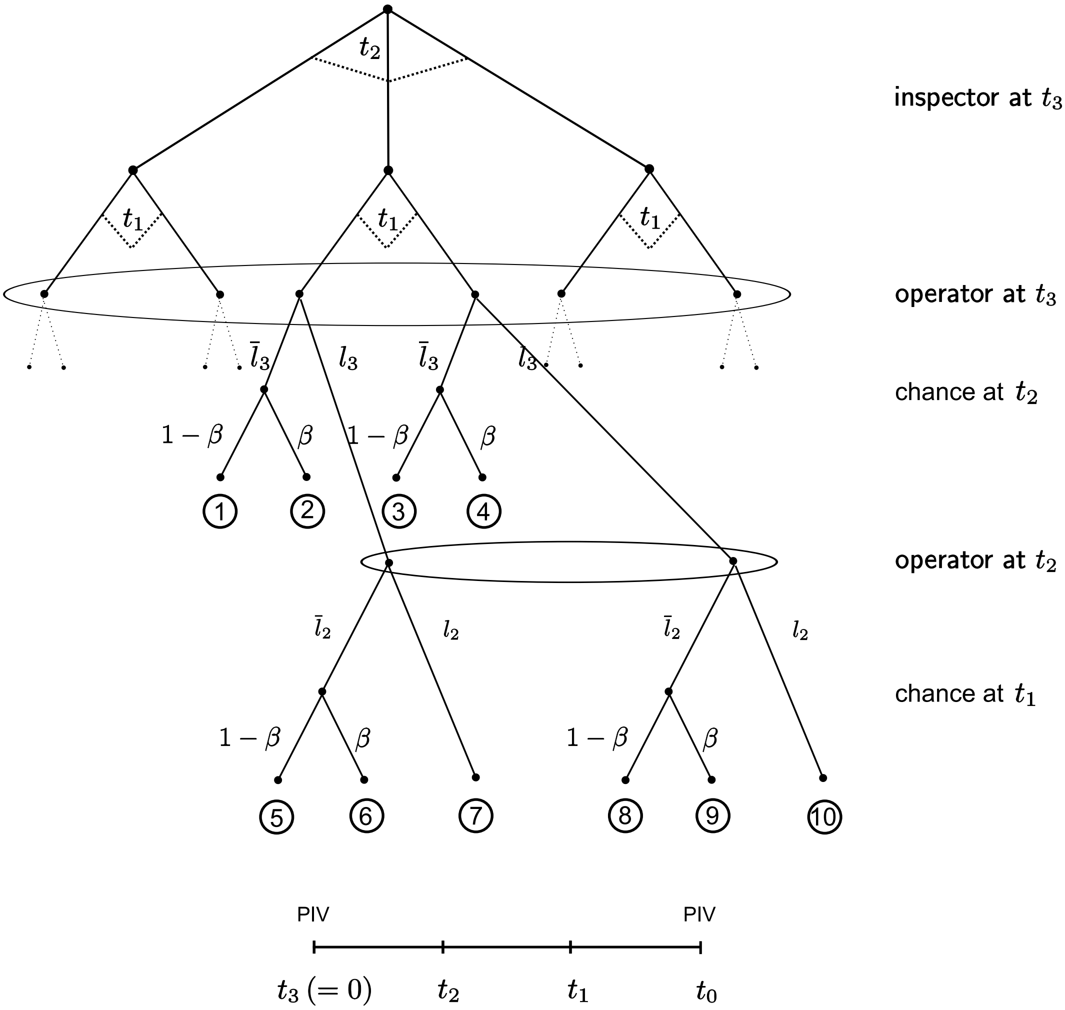


Figure 2: Graphical representation of the extensive form of the *sequential illegal* inspection game with two inspections during the reference time interval and without errors of the first and second kind, i.e., $\alpha = \beta = 0$. The box indicates a subgame.



① $\equiv t_2 - t_3 \equiv$ ③

② $\equiv (1-\beta)\cdot(t_1 - t_3) + \beta\cdot(t_0 - t_3) \equiv$ ④

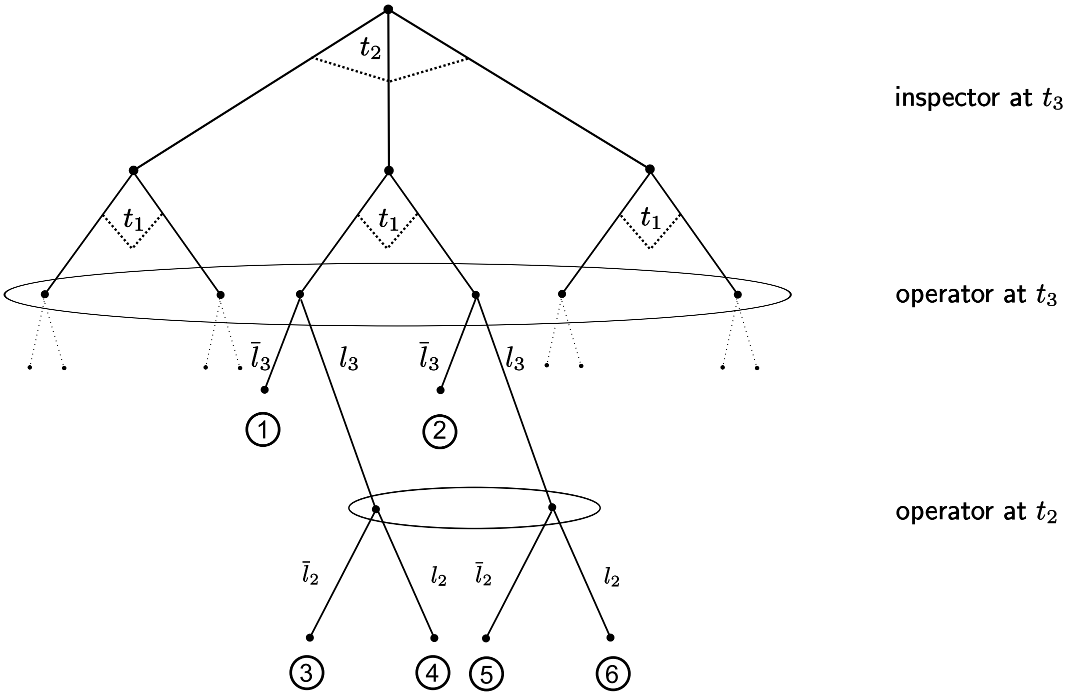
⑤ $\equiv t_1 - t_2 \equiv$ ⑧

⑥ $\equiv t_0 - t_2 \equiv$ ⑨

⑦ $\equiv t_0 - t_1 \equiv$ ⑩

of course for example ① and ③ refer to different values of t_1 , and so on.

Figure 3: Graphical representation of the extensive form of the *hybrid-sequential illegal* inspection game with two inspections and the possibility of an error of the second kind, i.e., $\alpha = 0$ and $\beta > 0$.



① $\equiv (1-\beta)\cdot(t_2-t_3)+\beta\cdot[\,(1-\beta)\cdot(t_1-t_3)+\beta\cdot(t_0-t_3)\,]\equiv$ ②

③ $\equiv (1-\beta)\cdot(t_1-t_2)+\beta\cdot(t_0-t_2)\equiv$ ⑤

④ $\equiv t_0-t_1\equiv$ ⑥

Figure 4: Graphical representation of the *reduced* extensive form of the game given in Figure 3.

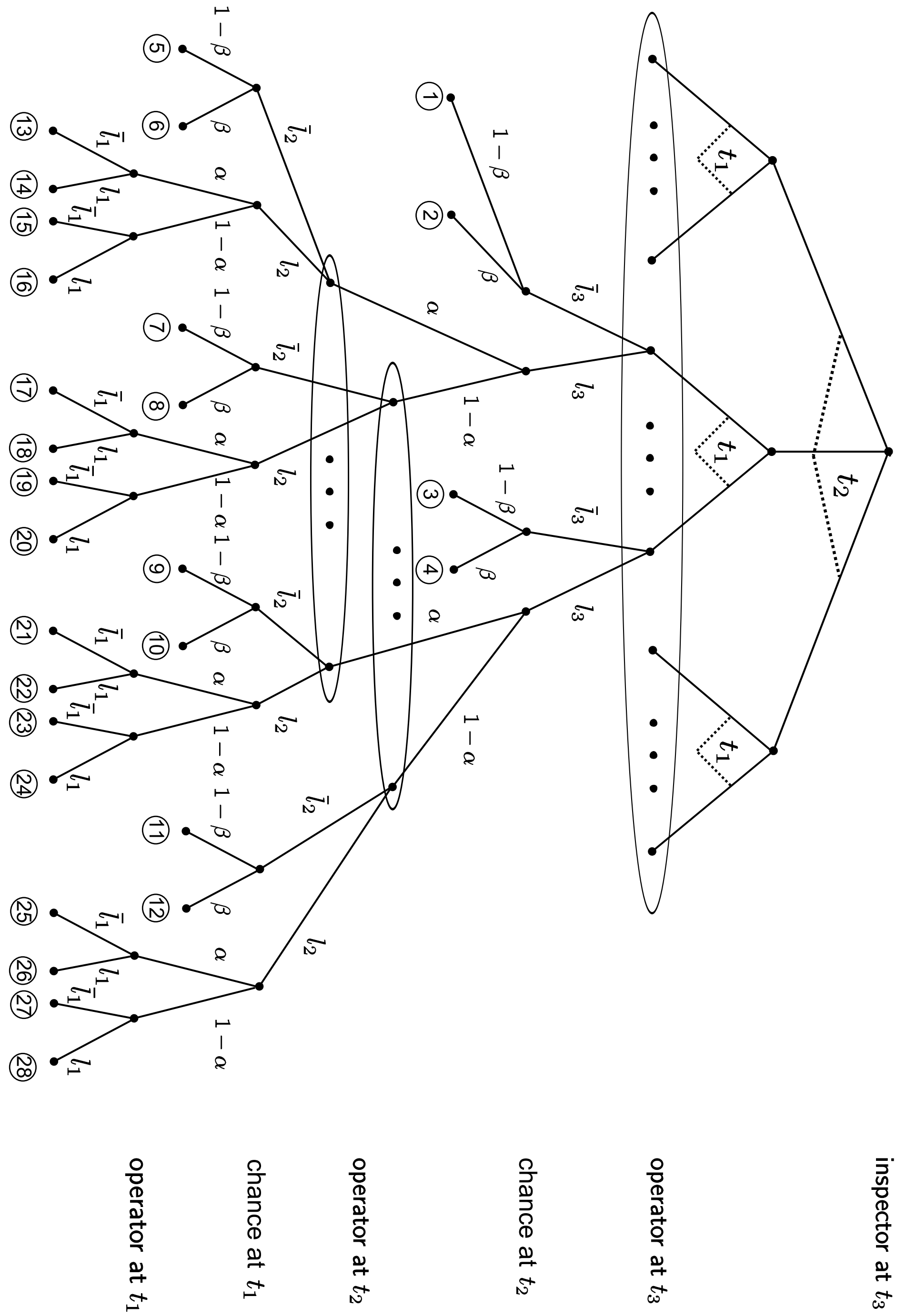


Figure 5: Graphical representation of the extensive form of our general *hybrid-sequential* inspection game with two inspections and the possibility of errors of the first and second kind, i.e., $\alpha > 0$ and $\beta > 0$. The payoffs to the two players at the end nodes are given in Figure 6.

The payoffs to the inspector are:

① $\equiv -a \cdot (t_2 - t_3) \equiv$ ③

② $\equiv -a \cdot [\,(1-\beta) \cdot (t_1 - t_3) + \beta \cdot (t_0 - t_3)\,] \equiv$ ④

⑤ $\equiv -e -\ a \cdot (t_1 - t_2) \equiv$ ⑨

⑥ $\equiv -e -\ a \cdot (t_0 - t_2) \equiv$ ⑩

⑦ $\equiv -a \cdot (t_1 - t_2) \equiv$ ⑪

⑧ $\equiv -a \cdot (t_0 - t_2) \equiv$ ⑫

⑬ $\equiv -2 \cdot e -\ a \cdot (t_0 - t_1) \equiv$ ㉑

⑭ $\equiv -2 \cdot e \equiv$ ㉒

⑮ $\equiv -e -\ a \cdot (t_0 - t_1) \equiv$ ⑰ $\equiv$ ㉓ $\equiv$ ㉕

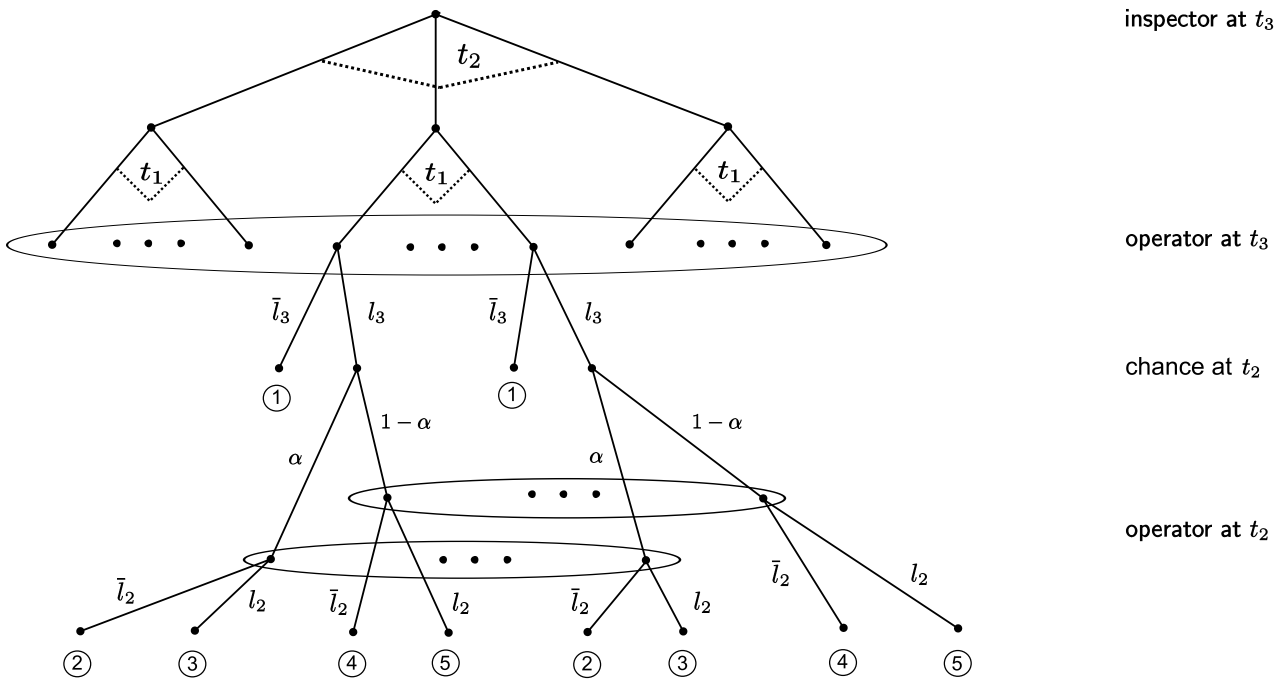
⑯ $\equiv -e \equiv$ ⑱ $\equiv$ ㉔ $\equiv$ ㉖

⑲ $\equiv -a \cdot (t_0 - t_1) \equiv$ ㉗

⑳ $\equiv 0 \equiv$ ㉘

The payoff to the operator is obtained from these payoffs by repacing - a by d, e by f and adding - b.

Figure 6: Payoffs to the two players for our general extensive form game given in Figure 5.



The payoffs to the inspector are:

(1) $\equiv -a \cdot \{(1-\beta)\cdot(t_2-t_3)+\beta\cdot[(1-\beta)\cdot(t_1-t_3)+\beta\cdot(t_0-t_3)]\}$

(2) $\equiv -e - a \cdot \{(1-\beta)\cdot(t_1-t_2)+\beta\cdot(t_0-t_2)\}$

(3) $\equiv \alpha \cdot (g_1 \cdot$ (13) $+(1-g_1)\cdot$ (14) $) + (1-\alpha)\cdot(g_1 \cdot$ (15) $+(1-g_1)\cdot$ (16) $)$

(4) $\equiv (1-\beta)\cdot(t_1-t_2)+\beta\cdot(t_0-t_2)$

(5) $\equiv \alpha \cdot (g_1 \cdot$ (17) $+(1-g_1)\cdot$ (18) $) + (1-\alpha)\cdot(g_1 \cdot$ (19) $+(1-g_1)\cdot$ (20) $)$

Here (13) -- (20) are taken from Figure 6.

The payoff to the operator is obtained from these payoffs by replacing - a by d, e by f and adding - b.

Figure 7: Reduced extensive form of the game given in Figure 5 and 6.

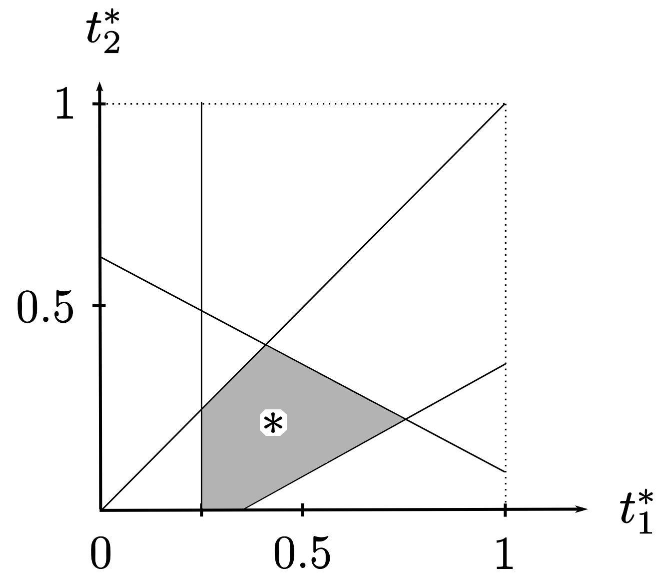


Figure 8: Graphical representation of the set of *legal equilibria* (t_2^*, t_1^*) of the inspector (shaded area) according to (8), i.e., equilibria in which the operator's equilibrium strategy is legal behavior. In the midst of this set the star indicates the equilibrium of the inspector according to (11).

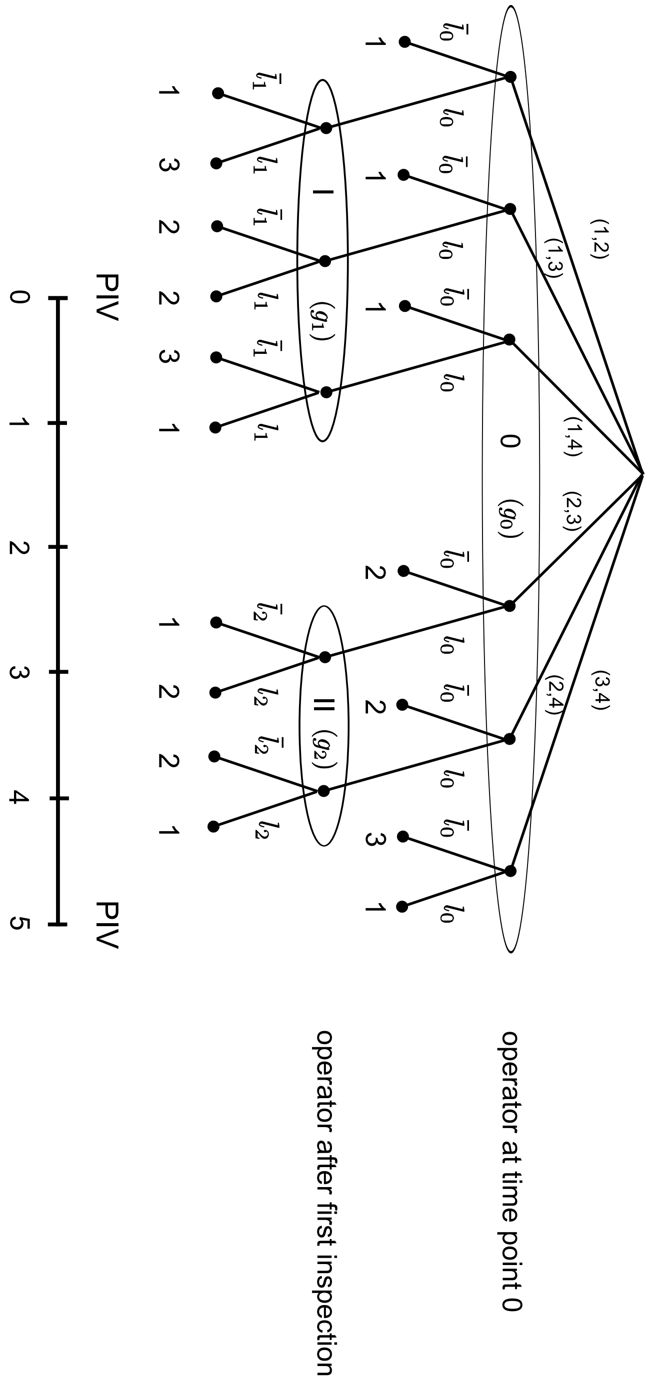


Figure 9: Graphical representation of the *discrete time* zero-sum *hybrid-sequential illegal* game with four possible time points for inspection and two inspections. Payoffs to the operator, as given at the end nodes, are the times between the start of the illegal activity and its detection.

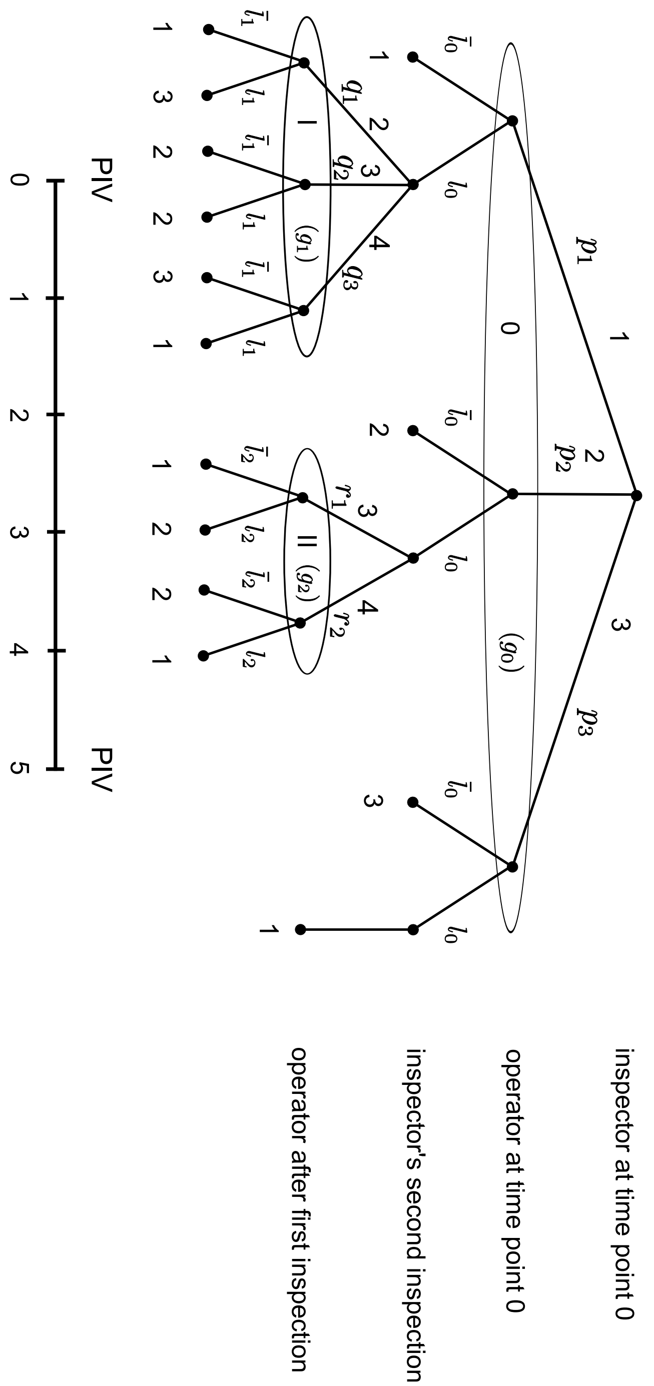


Figure 10: Graphical representation of the *discrete time* zero-sum *sequential illegal* game with four possible time points for inspection and two inspections. Payoffs to the operator, as given at the end nodes, are the times between the start of the illegal activity and its detection.