Universität der Bundeswehr München

FAKULTÄT FÜR INFORMATIK

On the Asymptotic Behavior of a Discrete Time Inspection Game: Proofs and Illustrations

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Bericht 2007-05
Januar 2008

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Abstract

In many material processing and storing plants an inspector performs during some reference time interval, e.g. one year, a number of inspections because it can not be excluded that the plant operator acts illegally by violating agreed rules, e.g., diverts valuable material. The inspections guarantee that any illegal action is detected at the earliest inspection following the beginning of that illegal action. We assume that the inspector wants to choose the time points for his inspections such that the time which elapses between the beginning of the illegal action and its detection is minimized whereas the operator wants to start his illegal action such that the elapsed time is maximized. Therefore, this inspection problem is modelled as a zero-sum game with strategies and payoffs as described.

Depending on the concrete situation the start of the illegal action and the inspections can take place either at a finite number of time points or at every time point of a reference time interval. The first case can be modelled as a zero-sum game with finite pure strategy sets while the latter one leads to a zero-sum game with infinite pure strategy sets and discontinuous payoff kernel.

The aim of this contribution is to demonstrate the close relation between both games for the case of one interim inspection.

On the Asymptotic Behavior of a Discrete Time Inspection Game: Proofs and Illustrations

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28th January 2008

1 Introduction

In many material processing and storing plants an inspector performs during some reference time interval, e.g. one year, a number of inspections because it can not be excluded that the plant operator acts illegally by violating agreed rules, e.g., diverts valuable material. The inspections guarantee that any illegal action is detected at the earliest inspection following the beginning of that illegal action. We assume that the inspector wants to choose the time points for his inspections such that the time which elapses between the beginning of the illegal action and its detection is minimized whereas the operator wants to start his illegal action such that the elapsed time is maximized. Therefore, this inspection problem is modelled as a zero-sum game with strategies and payoffs as described.

In reliability theory, variants of this game have been studied by Derman (see [Der61]). An operating unit may fail which creates costs that increase with the time until the failure is detected. The overall time interval represents the time between normal replacements of the unit. A pessimistic assumption about the way failures occur leads to a minimax analysis which is essentially the same as that considered in section 4.

Another application is the inspection of a nuclear or chemical plant subject to verification in the framework of an international arms control and disarmament treaty (see [ACvS91] or [IAE]). Nuclear plants are regularly inspected at the end of the year. If nuclear material is diverted form such a facility for non-peaceful purposes the inspector may wish to discover this not only at the end of the year but earlier which is the purpose of interim inspections. Since this application was the motivation for this chapter, in the following we only use the term *illegal action*, keeping in mind that in other applications also a failure could be meant.

In section 2 we will formalize the situation for one interim inspection with the help of a zero-sum game with finite pure strategy sets. This game is solved in section 3 and statements about the asymptotic behavior of strategies and payoff´s are made. In section 4 we introduce the corresponding continuous time inspection game and its solution and compare it with that from section 3.

2 The model

Let N be the number of possible time points for an inspection. The general inspection situation is depicted in Figure 1.

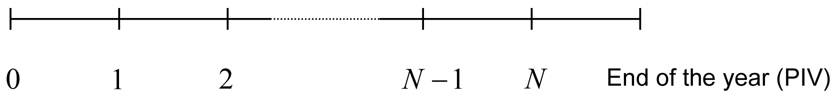

Figure 1: General inspection situation.

Our model works under the following assumptions:

- The operator decides at which of the possible time points $0, 1, \ldots, N$ he will start his illegal action.
- The inspector decides at which time point he will perform his inspection. At the beginning and the end of the reference time interval, e.g., one year, a physical inventory (PIV) is taken which specifies with certainty that plant operations were performed in compliance with - for instance the Non-Proliferation Treaty (NPT) - obligations. He can choose the inspection time point freely from the set $1, 2, \ldots, N$.
- Once an illegal action has been started by the operator, the inspector will detect it during the next intermediate inspection (if there is still one) or with certainty at the end of the year, i.e., the illegal action is discovered at the earliest inspection following the start of the illegal action.
- The players choose their strategies simultaneously at the beginning of the year. Depending on N the operator's payoff will be the elapsed time between start and detection of the illegal action. The payoff to the inspector will be the negative of the payoff to the operator.

Let $\Phi_{Op} = \{0, 1, \ldots, N\}$ and $\Phi_{Insp} = \{1, \ldots, N\}$ be the sets of pure strategies of the operator and the inspector. If i is the time point of the beginning of the illegal action of the operator and j the time point of the inspection, then we obtain – according to our model assumptions – for the payoff to the operator

$$a_{ij} \quad := \quad Op(i,j) = \begin{cases} j & : \; i = 0 \quad \text{and} \quad j = 1, \ldots, N \\ N - i + 1 & : \; i = 1, \ldots, N-1 \quad \text{and} \quad j = 1, \ldots, i \\ j - i & : \; i = 1, \ldots, N-1 \quad \text{and} \quad j = i+1, \ldots, N \\ 1 & : \; i = N \quad \text{and} \quad j = 1, \ldots, N \end{cases} . \qquad (1)$$

The payoff to the inspector is $Insp(i,j) := -Op(i,j)$, i.e., we are dealing with a zero-sum game. Matrix A with the entries a_{ij} as defined above is called payoff matrix of the game.

Let us conclude our model description with two remarks: First, we assume that if the times of inspection and start of illegal action coincide, i.e., $j = i$, the illegal action is not detected until the next inspection at the end of the reference time interval. Second, we deal only with the illegal inspection game, i.e., the game where legal behavior of the operator is a priori excluded. A short remark about legal behavior is made at the end of section 3.

3 Solution of the discrete time inspection game

The payoff matrix A of our discrete time inspection game is shown in Table 1. The pure strategies of the operator resp. the inspector are depicted the first column resp. the first row. The entries in the payoff matrix are the payoff's to the operator.

	1	2	3	$\cdots$	n	$n+1$	$\cdots$	$N-1$	N
0	1	2	3	$\cdots$	n	$n+1$	$\cdots$	$N-1$	N
1	N	1	2	$\cdots$	$n-1$	n	$\cdots$	$N-2$	$N-1$
2	$N-1$	$N-1$	1	$\cdots$	$n-2$	$n-1$	$\cdots$	$N-3$	$N-2$
$\vdots$				$\ddots$					
n	$N-n+1$	$N-n+1$	$N-n+1$	$\cdots$	$N-n+1$	1	$\cdots$	$N-n-1$	$N-n$
$n+1$	$N-n$	$N-n$	$N-n$	$\cdots$	$N-n$	$N-n$	$\cdots$	$N-n-2$	$N-n-1$
$\vdots$				$\vdots$					
$N-1$	2	2	2	$\cdots$	2	2	$\cdots$	2	1
N	1	1	1	$\cdots$	1	1	$\cdots$	1	1

Table 1: The payoff matrix A to the discrete time inspection game for arbitrary N.

We first want to answer the question, if there is a pure strategy combination which leads to a stable situation of the game, i.e., a pair of strategies from which no player has an incentive to deviate. The answer is no. Formally we are looking for a pure strategy combination (i^*, j^*) with the so-called saddle point property

$$Op(i, j^*) \leq Op(i^*, j^*) \leq Op(i^*, j) \tag{2}$$

for all $i = 0, 1, \ldots, N$ and $j = 1, \ldots, N$. The left hand inequality specifies the operator's gain of maximizing his payoff, while the right hand inequality specifies the inspector's gain of minimizing the elapsed time. Suppose there would be a stable situation (i^*, j^*) in pure strategies. Then we would obtain with (1)

$$Op(i^*, j^*) = \max_{i=0,\ldots,N} Op(i, j^*) \geq \frac{N+1}{2}$$

and

$$Op(i^*, j^*) = \min_{j=1,\ldots,N} Op(i^*, j) = 1\,,$$

i.e., (2) cannot be fulfilled for $N \geq 2$. This argumentation shows, that in our game no stable situation in pure strategies exists. Therefore, we have to introduce – following the general procedure in non-cooperative game theory – the concept of mixed strategy. A mixed strategy of a player is a probability distribution over his set of pure strategies, i.e., for the operator

$$Q_{Op} := \left\{ (q_0, q_1, \ldots, q_N)^T \in \mathbb{R}^{N+1} : q_i \geq 0 \quad \text{for} \quad i = 0, \ldots, N \quad \text{and} \quad \sum_{i=0}^{N} q_i = 1 \right\}$$

and for the inspector

$$Q_{Insp} := \left\{ (p_1, \ldots, p_N)^T \in \mathbb{R}^{N} : p_j \geq 0 \quad \text{for} \quad j = 1, \ldots, N \quad \text{and} \quad \sum_{j=1}^{N} p_j = 1 \right\}.$$

The i-th resp. the j-th pure strategy of the operator resp. the inspector corresponds to the $(i+1)$-th resp. the j-th unit vector. In order to avoid problems with the enumeration we will write - although mathematical slightly incorrect - i instead of e_{i+1} and j instead of e_j. If the players decide to play the mixed strategy combination $(\mathbf{q}, \mathbf{p})$, the operator's expected payoff defined on the set $Q_{Op} \times Q_{Insp}$ is given by

$$Op(\mathbf{q}, \mathbf{p}) := \mathbf{q}^T A \, \mathbf{p} = \sum_{i=0}^{N} \sum_{j=1}^{N} q_i \, p_j \, Op(i, j) \,. \tag{3}$$

According to our assumptions the inspector's expected payoff is $Insp(\mathbf{q}, \mathbf{p}) = -Op(\mathbf{q}, \mathbf{p})$.

Now the idea of the stable situation from the discussion above can be generalized to the saddle point criterion (see, e.g., [Mye97]):

Definition 1. *A mixed strategy combination $(\mathbf{q}^*, \mathbf{p}^*) \in Q_{Op} \times Q_{Insp}$ constitutes a saddle point in mixed strategies of the zero-sum game with payoff matrix A if and only if*

$$Op(\mathbf{q}, \mathbf{p}^*) \leq Op(\mathbf{q}^*, \mathbf{p}^*) \leq Op(\mathbf{q}^*, \mathbf{p}) \quad \textit{for all} \quad \mathbf{q} \in Q_{Op} \quad \textit{and all} \quad \mathbf{p} \in Q_{Insp} \,,$$

where $Op(\mathbf{q}, \mathbf{p})$ is given by (3). □

It can be shown that $(\mathbf{q}^*, \mathbf{p}^*)$ is a saddle point of the zero-sum game if only if

$$Op(i, \mathbf{p}^*) \leq Op(\mathbf{q}^*, \mathbf{p}^*) \leq Op(\mathbf{q}^*, j) \quad \text{for all} \quad i = 0, \ldots, N \quad \text{and all} \quad j = 1, \ldots, N \,, \tag{4}$$

see, e.g., [Mye97], i.e., both inequalities have only to be proven for the pure strategies of the players.

$Op(\mathbf{q}^*, \mathbf{p}^*)$ is called the value of the game. It can be shown that every zero-sum game with finite pure strategy sets possesses at least on saddle point in mixed strategies (see [vNM47] or [Nas51]), but of course - see the argumentation above - not always a saddle point in pure strategy combinations. If a zero-sum game has the saddle points $(\mathbf{q}^*, \mathbf{p}^*)$ and $(\mathbf{q}_1^*, \mathbf{p}_1^*)$, then $(\mathbf{q}^*, \mathbf{p}_1^*)$ and $(\mathbf{q}_1^*, \mathbf{p}^*)$ are also saddle points of the game with the property

$$Op(\mathbf{q}^*, \mathbf{p}^*) = Op(\mathbf{q}^*, \mathbf{p}_1^*) = Op(\mathbf{q}_1^*, \mathbf{p}^*) = Op(\mathbf{q}_1^*, \mathbf{p}_1^*) \,,$$

i.e., all saddle points are interchangeable and lead to the same value. For this reason finding all saddle points is more a mathematical challenge than necessary for applications.

In the next Lemma we state a recursive relation between $Op(\mathbf{q}, j+1)$ and $Op(\mathbf{q}, j)$ as well as $Op(i+1, \mathbf{p})$ and $Op(i, \mathbf{p})$, which we will use in the proof of Theorem 1.

Lemma 1. *Consider the zero-sum game with payoff matrix A given by (1). Then for all $\mathbf{q} = (q_0, q_1, \ldots, q_N)^T \in Q_{Op}$ and $\mathbf{p} = (p_1, \ldots, p_N)^T \in Q_{Insp}$ the following recursive relations hold:*

$$Op(\mathbf{q}, j+1) = Op(\mathbf{q}, j) - (N - j + 1) \cdot q_j + \sum_{i=0}^{j} q_i \quad \textit{for all} \quad j \in \{1, \ldots, N-1\} \tag{5}$$

and

$$Op(i+1, \mathbf{p}) = Op(i, \mathbf{p}) + (N - i) \cdot p_{i+1} - 1 \quad \textit{for all} \quad i \in \{0, \ldots, N-1\} \,. \tag{6}$$

Proof. Formula (1) may also be written as

$$a_{ij} \quad = \quad \begin{cases} j-i & : \quad i=0,\ldots,j-1 \\ N-i+1 & : \quad i=j,\ldots,N \end{cases} \tag{7}$$

for all $j = 1, \ldots, N$. With (7) we get for all $j \in \{1, \ldots, N\}$ and all $\mathbf{q} = (q_0, q_1, \ldots, q_N)^T \in Q_{Op}$

$$Op(\mathbf{q}, j) = \sum_{i=0}^{j-1} (j-i) \cdot q_i + \sum_{i=j}^{N} (N-i+1) \cdot q_i \,. \tag{8}$$

Let us fix an index $j \in \{1, \ldots, N-1\}$ and a mixed strategy $\mathbf{q} \in Q_{Op}$. Then we obtain with (8)

$$\begin{aligned} Op(\mathbf{q}, j+1) &= \sum_{i=0}^{j} (j+1-i) \cdot q_i + \sum_{i=j+1}^{N} (N-i+1) \cdot q_i \\ &= \sum_{i=0}^{j} (j-i) \cdot q_i + \sum_{i=0}^{j} q_i + \sum_{i=j}^{N} (N-i+1) \cdot q_i - (N-j+1) \cdot q_j \\ &= Op(\mathbf{q}, j) - (N-j+1) \cdot q_j + \sum_{i=0}^{j} q_i \,, \end{aligned}$$

i.e., recursive relation (5). For the proof of the second relation we first get from (1) for all $\mathbf{p} = (p_1, \ldots, p_N)^T \in Q_{Insp}$

$$Op(i, \mathbf{p}) = \begin{cases} \displaystyle\sum_{j=1}^{N} j \cdot p_j & i = 0 \\ \displaystyle (N-i+1) \cdot \sum_{j=1}^{i} p_j + \sum_{j=i+1}^{N} (j-i) \cdot p_j & i = 1, \ldots, N-1 \\ 1 & i = N \end{cases} . \tag{9}$$

With (9) we obtain

$$Op(1, \mathbf{p}) \quad = \quad N \cdot p_1 + \sum_{j=2}^{N} (j-1) \cdot p_j = Op(0, \mathbf{p}) + N \cdot p_1 - 1 \,,$$

and

$$Op(N, \mathbf{p}) \quad = \quad 1 = \sum_{j=1}^{N} p_j = 2 \cdot \sum_{j=1}^{N-1} p_j + 2 \cdot p_N - 1 = Op(N-1, \mathbf{p}) + p_N - 1 \,,$$

i.e., equation (6) for $i = 0$ and $i = N-1$. For a fixed index $i \in \{1, \ldots, N-2\}$ we get again from (9)

$$\begin{aligned} Op(i+1, \mathbf{p}) &= (N-i) \cdot \sum_{j=1}^{i+1} p_j + \sum_{j=i+2}^{N} (j-(i+1)) \cdot p_j \\ &= (N-i+1) \cdot \sum_{j=1}^{i} p_j + (N-i) \cdot p_{i+1} - \sum_{j=1}^{i} p_j + \sum_{j=i+1}^{N} (j-i) \cdot p_j - \sum_{j=i+1}^{N} p_j \\ &= Op(i, \mathbf{p}) + (N-i) \cdot p_{i+1} - 1 \,, \end{aligned}$$

i.e., equation (6) for $i \in \{1, \dots, N-2\}$, which completes the proof. $\square$

The solution of our game is presented as Theorem 1.

Theorem 1. *Consider the zero-sum game with payoff matrix A given by (1). For $N \geq 2$ we define the cutting edge*

$$n^* = n^*(N) := \min \left\{ n : n \in \{1, \dots, N\} \quad \text{with} \quad \sum_{j=1}^{n} \frac{1}{N-j+1} \geq 1 \right\} \tag{10}$$

and therewith

$$q_i^* = q_i^*(N) := \begin{cases} \frac{1}{N} \cdot (N - n^* + 1) & : \quad i = 0 \\ \frac{(N - n^* + 1)}{(N-i+1) \cdot (N-i)} & : \quad i = 1, \dots, n^* - 1 \\ 0 & : \quad i = n^*, \dots, N \end{cases} \tag{11}$$

and

$$p_j^* = p_j^*(N) := \begin{cases} \frac{1}{N-j+1} & : \quad j = 1, \dots, n^* - 1 \\ 1 - \sum_{j=1}^{n^*-1} \frac{1}{N-j+1} & : \quad j = n^* \\ 0 & : \quad j = n^* + 1, \dots, N \end{cases} . \tag{12}$$

Then $(\mathbf{q}^*, \mathbf{p}^*)$ with $\mathbf{q}^* = (q_0^*, q_1^*, \dots, q_N^*)^T$ and $\mathbf{p}^* = (p_1^*, \dots, p_N^*)^T$ is a saddle point of the game with the value

$$Op_N^* := Op(\mathbf{q}^*, \mathbf{p}^*) = (N - n^* + 1) \cdot \sum_{j=1}^{n^*-1} \frac{1}{N-j+1} + 1 \, . \tag{13}$$

Proof. The proof is presented in several steps.

1. From (11) and (12) it can be directly seen that the components of $\mathbf{q}^*$ and $\mathbf{p}^*$ are greater or equal to 0 and that both vectors are correctly normalized, i.e., $\mathbf{q}^*$ and $\mathbf{p}^*$ are probability distributions over Φ_{Op} resp. Φ_{Insp}.

2. Saddle point inequalities: For a fixed index $j \in \{1, \dots, n^* - 1\}$ we have with (11)

$$\sum_{i=0}^{j} q_i^* \; = \; (N - n^* + 1) \left(\frac{1}{N} + \sum_{i=1}^{j} \left(\frac{1}{N-i} - \frac{1}{N-i+1} \right) \right) = (N - n^* + 1) \cdot \frac{1}{N-j}$$

and hence

$$\sum_{i=0}^{j} q_i^* = \begin{cases} (N - n^* + 1) \cdot \frac{1}{N-j} = (N - j + 1) \cdot q_j^* & : \quad j = 1, \dots, n^* - 1 \\ 1 & : \quad j = n^*, \dots, N \end{cases} . \tag{14}$$

This leads together with recursive relation (5) to

$$Op(\mathbf{q}^*, 1) = Op(\mathbf{q}^*, 2) = \ldots = Op(\mathbf{q}^*, n^*) \tag{15}$$

and

$$Op(\mathbf{q}^*, N) > Op(\mathbf{q}^*, N-1) > \ldots > Op(\mathbf{q}^*, n^*+1) > Op(\mathbf{q}^*, n^*)\,. \tag{16}$$

On the other hand we obtain with (6) and (12)

$$Op(0, \mathbf{p}^*) = Op(1, \mathbf{p}^*) = \ldots = Op(n^*-1, \mathbf{p}^*) \tag{17}$$

and

$$Op(N, \mathbf{p}^*) < Op(N-1, \mathbf{p}^*) < \ldots < Op(n^*-1, \mathbf{p}^*)\,. \tag{18}$$

Combining (15) and (17) gives us

$$\begin{aligned} Op(\mathbf{q}^*, \mathbf{p}^*) &= (\mathbf{q}^*)^T A\, \mathbf{p}^* \\ &= Op(\mathbf{q}^*, 1) = Op(\mathbf{q}^*, 2) = \ldots = Op(\mathbf{q}^*, n^*) \\ &= Op(0, \mathbf{p}^*) = Op(1, \mathbf{p}^*) = \ldots = Op(n^*-1, \mathbf{p}^*)\,. \end{aligned} \tag{19}$$

Now, Formulae (15) - (18) together with (19) implies

$$Op(i, \mathbf{p}^*) \le Op(\mathbf{q}^*, \mathbf{p}^*) \le Op(\mathbf{q}^*, j) \quad \text{for all} \quad i = 0, \ldots, N \quad \text{and} \quad j = 1, \ldots, N\,.$$

Making use of (4) we see that $(\mathbf{q}^*, \mathbf{p}^*)$ is a saddle point of the game.

3. The value of the game: With (1) and (19) we get

$$Op(\mathbf{q}^*, \mathbf{p}^*) = Op(n^*-1, \mathbf{p}^*) = (N - n^* + 2) \sum_{j=1}^{n^*-1} p_j^* + p_{n^*}^* = (N - n^* + 1) \sum_{j=1}^{n^*-1} p_j^* + 1\,, \tag{20}$$

i.e., (13) as required. □

The saddle point $(\mathbf{q}^*, \mathbf{p}^*)$ has an interesting property: We see that the pure strategies $n^*, \ldots, N$ for the operator and $n^*+1, \ldots, N$ for the inspector are cut off and are never played in the saddle point. That means that the operator will never perform an illegal action after time point n^* and the inspector will never inspect after time point n^*+1. This makes sense since detection is guaranteed to occur at the end of the reference time interval and the operator will not wish to wait too long before violating.

In Figure 2 the optimal strategies of both players are depicted for the case of $N = 19$. For arbitrary $N \ge 3$ we obtain from (11)

$$q_0^*(N) > q_1^*(N) \quad \text{and} \quad q_1^*(N) < q_2^*(N) < \ldots < q_{n^*-1}^*(N)$$

and from (12)

$$p_1^*(N) < p_2^*(N) < \ldots < p_{n^*-1}^*(N)\,.$$

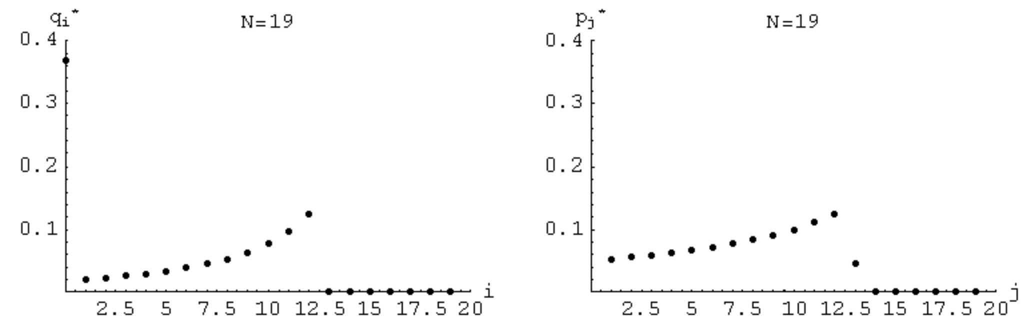

Figure 2: The optimal strategies $\mathbf{q}^*$ and $\mathbf{p}^*$ for $N = 19$.

In general nothing can be said about the ratio between $p^*_{n^*}$ and p^*_j $(j = 1, \ldots, n^* - 1)$: In case of $N = 4$ we have $n^*(4) = 3$ and

$$p^*_1 = 1/4\,, p^*_2 = 1/3\,, p^*_3 = 5/12\,, p^*_4 = 0\,, \quad \text{i.e.,} \quad p^*_1 < p^*_2 < p^*_3\,,$$

while in case of $N = 5$ we get $n^*(5) = 4$ and

$$p^*_1 = 1/5\,, p^*_2 = 1/4\,, p^*_3 = 1/3\,, p^*_4 = 13/60\,, p^*_5 = 0\,, \quad \text{i.e.,} \quad p^*_1 < p^*_4 < p^*_2 < p^*_3\,.$$

Formulae (10) and (13) can hardly be used in order to get ideas about the orders of magnitude of $n^*(N)$ and Op^*_N. Therefore we present in the next Lemma lower and upper bounds for these quantities.

Lemma 2. *Consider the zero-sum game with payoff matrix A given by (1). Then we obtain for the cutting edge $n^*(N)$*

$$\left(1 - \frac{1}{e}\right) N < n^*(N) < \left(1 - \frac{1}{e}\right) (N + 1) + 1 \tag{21}$$

and for the value of the game Op^*_N

$$(N - n^* + 1) < Op^*_N < (N - n^* + 2)\,. \tag{22}$$

Proof. For an arbitrary number $n \in \{1, \ldots, N - 1\}$ we first have

$$\int_1^{n+1} \frac{1}{N - x + 2}\, dx \leq \sum_{j=1}^{n} \frac{1}{N - j + 1} \leq \int_1^{n+1} \frac{1}{N - x + 1}\, dx$$

and therewith

$$\ln\left[\frac{N + 1}{N - n + 1}\right] \leq \sum_{j=1}^{n} \frac{1}{N - j + 1} \leq \ln\left[\frac{N}{N - n}\right]. \tag{23}$$

From (10) we obtain the inequalities

$$\sum_{j=1}^{n^*} \frac{1}{N - j + 1} \geq 1 \quad \text{and} \quad \sum_{j=1}^{n^*-1} \frac{1}{N - j + 1} < 1\,.$$

This leads with (23) to

$$\ln\left[\frac{N}{N-n^*}\right] \geq 1 \quad \text{and} \quad \ln\left[\frac{N+1}{N-n^*+2}\right] < 1\,.$$

Combining both inequalities we get

$$\left(1-\frac{1}{e}\right) N \leq n^* < \left(1-\frac{1}{e}\right)(N+1)+1\,.$$

Since $n^*(N)$ is a natural number, the left hand inequality is indeed strict. Now we prove inequality (22). From (18) we get

$$Op_N^* = Op(n^*-1,\mathbf{p}^*) > Op(n^*,\mathbf{p}^*) = (N-n^*+1)\,,$$

whereas with (1) and (20) we obtain

$$\begin{aligned} Op_N^* &= Op(n^*-1,\mathbf{p}^*) = (N-n^*+2)\cdot\sum_{j=1}^{n^*-1} p_j^* + p_{n^*}^* = (N-n^*+2)\cdot(1-p_{n^*}^*) + p_{n^*}^* \\ &< (N-n^*+2)\,, \end{aligned}$$

i.e., (22), as required. □

In order to get an idea of the behavior of the cutting edge $n^*(N)$ and the value of the game Op_N^*, we present in Table 2 these quantities as well as the corresponding normalized quantities $\frac{1}{N+1}\,n^*(N)$ and $\frac{1}{N+1}Op_N^*$. It can be seen, that the normalized cutting edge $\frac{1}{N+1}\,n^*(N)$ is neither an increasing nor a decreasing function of N. In the next Lemma we show, that $n^*(N)$ and Op_N^* are increasing functions of N, while $\frac{1}{N+1}\,Op_N^*$ is a decreasing function of N.

Lemma 3. *Consider the zero-sum game with payoff matrix A given by (1). Then for the quantities $n^*(N)$, Op_N^* and $\frac{1}{N+1}\,Op_N^*$ the following inequalities hold:*

$$n^*(N) \leq n^*(N+1) \leq n^*(N)+1$$

as well as

$$Op_N^* < Op_{N+1}^* \qquad \text{and} \qquad \frac{1}{N+1}\,Op_N^* \geq \frac{1}{N+2}\,Op_{N+1}^*\,.$$

Proof. The proof is presented in several steps.

1. We first proof the inequality $n^*(N) \leq n^*(N+1)$. Definition (10) applied to $n^*(N+1)$ leads to

$$\sum_{j=1}^{n^*(N+1)} \frac{1}{(N+1)-j+1} \geq 1$$

and therewith to

$$\sum_{j=1}^{n^*(N+1)-1} \frac{1}{N-j+1} \geq 1 - \frac{1}{N+1}\,. \tag{24}$$

N	$n^*(N)$	$\frac{1}{N+1}\, n^*(N)$	Op_N^*	$\frac{1}{N+1}\, Op_N^*$
2	2	0.666667	1.5	0.5
3	3	0.75	1.83333	0.458333
4	3	0.6	2.16667	0.433333
5	4	0.666667	2.56667	0.427778
6	5	0.714286	2.9	0.414286
7	5	0.625	3.27857	0.409821
8	6	0.666667	3.65357	0.405952
10	7	0.636364	4.38254	0.398413
12	8	0.615385	5.09939	0.392261
13	9	0.642857	5.484	0.391714
14	10	0.666667	5.84114	0.38941
20	13	0.619048	8.03906	0.382812
30	20	0.645161	11.7262	0.378265
40	26	0.634146	15.4047	0.375725
100	64	0.633663	37.4743	0.371032

Table 2: Behavior of the cutting edge $n^*(N)$, the value of the game Op_N^* and its normalized values relative to $N+1$ (rounded).

Suppose there would be a natural number N with $n^*(N) > n^*(N+1)$. Then we obtain with (10) applied to $n^*(N)$ and $n^*(N+1)$ and (24)

$$\begin{aligned} 1 > \sum_{j=1}^{n^*(N)-1} \frac{1}{N-j+1} &= \sum_{j=1}^{n^*(N+1)-1} \frac{1}{N-j+1} + \sum_{j=n^*(N+1)}^{n^*(N)-1} \frac{1}{N-j+1} \\ &\geq 1 - \frac{1}{N+1} + \sum_{j=n^*(N+1)}^{n^*(N)-1} \frac{1}{N-j+1}\,, \end{aligned}$$

which implies

$$\frac{1}{N+1} > \sum_{j=n^*(N+1)}^{n^*(N)-1} \frac{1}{N-j+1}\,.$$

This inequality cannot be true, since even for $j = n^*(N+1)$ we have $N - n^*(N+1) + 1 < N+1$. So we conclude that $n^*(N) \leq n^*(N+1)$ for all $N \geq 2$. The right hand inequality can be deduced as follows. Again we obtain with (10)

$$\sum_{j=1}^{n^*(N)} \frac{1}{N-j+1} \geq 1$$

and therewith

$$\sum_{j=1}^{n^*(N)} \frac{1}{(N+1)-j+1} \geq 1 + \frac{1}{N+1} - \frac{1}{N-n^*(N)+1}\,. \tag{25}$$

Let us suppose again that there exists a natural number N with $n^*(N+1) > n^*(N)+1$. This leads with (10) applied to $n^*(N+1)$ and (25) to

$$
\begin{aligned}
1 &> \sum_{j=1}^{n^*(N+1)-1} \frac{1}{(N+1)-j+1} = \sum_{j=1}^{n^*(N)} \frac{1}{(N+1)-j+1} + \sum_{j=n^*(N)+1}^{n^*(N+1)-1} \frac{1}{(N+1)-j+1} \\
&\geq 1 + \frac{1}{N+1} - \frac{1}{N-n^*(N)+1} + \sum_{j=n^*(N)+1}^{n^*(N+1)-1} \frac{1}{(N+1)-j+1}\,,
\end{aligned}
$$

which implies

$$
\frac{1}{N-n^*(N)+1} - \frac{1}{N+1} > \sum_{j=n^*(N)+1}^{n^*(N+1)-1} \frac{1}{(N+1)-j+1}\,.
$$

In case of $n^*(N)+1 = n^*(N+1)-1$ resp. $n^*(N)+1 < n^*(N+1)-1$ we obtain the inequalities

$$
-\frac{1}{N+1} > 0 \quad \text{resp.} \quad -\frac{1}{N+1} > \sum_{j=n^*(N)+2}^{n^*(N+1)-1} \frac{1}{(N+1)-j+1}\,,
$$

which both cannot be fulfilled. So we conclude that $n^*(N+1) \leq n^*(N)+1$ for all $N \geq 2$.

2. Now we proof the inequality $Op_N^* < Op_{N+1}^*$ and distinguish the two cases $n^*(N+1) = n^*(N)$ and $n^*(N+1) = n^*(N)+1$. The optimal strategies of the inspector for both cases are shown in Table 3 and Table 4. In order to simplify the following equations we write instead of $n^*(N)$ simply n^*.

j	1	2	$\cdots$	n^*-1	n^*
$p_j^*(N)$	$\frac{1}{N}$	$\frac{1}{N-1}$	$\cdots$	$\frac{1}{N-n^*+2}$	$1-\sum_{j=1}^{n^*-1}\frac{1}{N-j+1}$
$p_j^*(N+1)$	$\frac{1}{N+1}$	$\frac{1}{N}$	$\cdots$	$\frac{1}{N-n^*+3}$	$1-\sum_{j=1}^{n^*-1}\frac{1}{(N+1)-j+1}$

Table 3: The optimal strategies of the inspector in case of $n^*(N+1) = n^*$.

j	1	2	$\cdots$	n^*-1	n^*	n^*+1
$p_j^*(N)$	$\frac{1}{N}$	$\frac{1}{N-1}$	$\cdots$	$\frac{1}{N-n^*+2}$	$1-\sum_{j=1}^{n^*-1}\frac{1}{N-j+1}$	0
$p_j^*(N+1)$	$\frac{1}{N+1}$	$\frac{1}{N}$	$\cdots$	$\frac{1}{N-n^*+3}$	$\frac{1}{N-n^*+2}$	$1-\sum_{j=1}^{n^*}\frac{1}{(N+1)-j+1}$

Table 4: The optimal strategies of the inspector in case of $n^*(N+1) = n^*+1$.

With Table 3 we get in case of $n^*(N+1) = n^*$

$$p^*_{n^*}(N+1) - p^*_{n^*}(N) = \frac{1}{N-n^*+2} - \frac{1}{N+1} \tag{26}$$

and with Table 4 in case of $n^*(N+1) = n^*+1$

$$p^*_{n^*+1}(N+1) - p^*_{n^*}(N) = -\frac{1}{N+1}\,. \tag{27}$$

Now we obtain with Table 3 and (26) in case of $n^*(N+1) = n^*$

$$\begin{aligned} Op^*_{N+1} - Op^*_N &= \sum_{j=1}^{N+1} j \cdot p^*_j(N+1) - \sum_{j=1}^{N} j \cdot p^*_j(N) \\ &= \sum_{j=1}^{n^*-1} \frac{1}{(N+1)-j+1} - \frac{n^*-1}{N-n^*+2} + n^* \cdot \left(\frac{1}{N-n^*+2} - \frac{1}{N+1}\right) \\ &= \underbrace{\sum_{j=1}^{n^*} \frac{1}{(N+1)-j+1}}_{\geq\ 1\ \text{from}\ (10)} - \underbrace{\frac{n^*}{N+1}}_{<\,1} > 0 \end{aligned}$$

and with Table 4 and (27) in case of $n^*(N+1) = n^*+1$

$$\begin{aligned} Op^*_{N+1} - Op^*_N &= \sum_{j=1}^{n^*} \frac{1}{(N+1)-j+1} - n^* \cdot p^*_{n^*}(N) + (n^*+1) \cdot p^*_{n^*+1}(N+1) \\ &= \sum_{j=1}^{n^*} \underbrace{\frac{1}{(N+1)-j+1}}_{\geq\ \frac{1}{N+1}} - \frac{n^*}{N+1} + p^*_{n^*+1}(N+1) \\ &\geq p^*_{n^*+1}(N+1) > 0\,. \end{aligned}$$

This completes the proof of the inequality $Op^*_N < Op^*_{N+1}$ for all $N \geq 2$.

3. We proof the inequality $(N+1) \cdot Op^*_{N+1} \leq (N+2) \cdot Op^*_N$ for all $N \geq 2$ with $n^*(N+1) = n^*(N) = n^*$ and start with some preliminary considerations. With Table 3 we obtain

$$\begin{aligned} (N+1) \cdot p^*_{n^*}(N+1) \quad &- \quad (N+2) \cdot p^*_{n^*}(N) \\ &= (N+1) - \underbrace{\sum_{j=1}^{n^*-1} \frac{(N+1)}{(N+1)-j+1}}_{=1+\sum_{j=1}^{n^*-2} \frac{(N+1)}{N-j+1}} - (N+2) + \sum_{j=1}^{n^*-1} \frac{(N+2)}{N-j+1} \\ &= -2 + \sum_{j=1}^{n^*-1} \frac{1}{N-j+1} + \frac{N+1}{N-n^*+2}\,. \end{aligned} \tag{28}$$

Furthermore we see that

$$\begin{aligned}
\sum_{j=1}^{n^*-1} j\cdot\frac{(N+1)}{(N+1)-j+1} \quad &- \quad \sum_{j=1}^{n^*-1} j\cdot\frac{(N+2)}{N-j+1} \\
&= \quad 1+\sum_{j=1}^{n^*-2}(j+1)\cdot\frac{(N+1)}{N-j+1}-\sum_{j=1}^{n^*-1} j\cdot\frac{(N+2)}{N-j+1} \\
&= \quad 1+\sum_{j=1}^{n^*-2}\underbrace{\frac{(j+1)\cdot(N+1)-j\cdot(N+2)}{N-j+1}}_{=1}-\frac{(n^*-1)\cdot(N+2)}{N-n^*+2} \\
&= \quad n^*-1-\frac{(n^*-1)\cdot(N+2)}{N-n^*+2}\,. \qquad (29)
\end{aligned}$$

Finally receive with Table 3, (28) and (29)

$$\begin{aligned}
(N+1)\cdot Op^*_{N+1} \quad &- \quad (N+2)\cdot Op^*_N \\
&= \quad \sum_{j=1}^{n^*-1} j\cdot\frac{(N+1)}{(N+1)-j+1}+n^*\cdot(N+1)\cdot p^*_{n^*}(N+1) \\
&- \quad \sum_{j=1}^{n^*-1} j\cdot\frac{(N+2)}{N-j+1}-n^*\cdot(N+2)\cdot p^*_{n^*}(N) \\
&= \quad n^*-1-\frac{(n^*-1)\cdot(N+2)}{N-n^*+2}+n^*\cdot\left(-2+\sum_{j=1}^{n^*-1}\frac{1}{N-j+1}+\frac{N+1}{N-n^*+2}\right) \\
&= \quad -n^*+\underbrace{n^*\cdot\sum_{j=1}^{n^*-1}\frac{1}{N-j+1}}_{<\ n^* \text{ from } (10)}<0\,,
\end{aligned}$$

as required. It should be noted that in this case (i.e., $n^*(N+1)=n^*$) even the strict inequality $(N+1)\cdot Op^*_{N+1}<(N+2)\cdot Op^*_N$ holds.

4. Now we proof the inequality $(N+1)\cdot Op^*_{N+1}\le(N+2)\cdot Op^*_N$ for all $N\ge 2$ with $n^*(N+1)=n^*(N)+1=n^*+1$ and start again with some preliminary considerations. With Table 4 we get

$$\begin{aligned}
(N+1)\cdot p^*_{n^*+1}(N+1) \quad &- \quad (N+2)\cdot p^*_{n^*}(N) \\
&= \quad (N+1)-\underbrace{\sum_{j=1}^{n^*}\frac{(N+1)}{(N+1)-j+1}}_{=1+\sum_{j=1}^{n^*-1}\frac{(N+1)}{N-j+1}}-(N+2)+\sum_{j=1}^{n^*-1}\frac{(N+2)}{N-j+1} \\
&= \quad -2+\sum_{j=1}^{n^*-1}\frac{1}{N-j+1}\,. \qquad (30)
\end{aligned}$$

Furthermore we obtain

$$\begin{aligned}
\sum_{j=1}^{n^*} j \cdot \frac{(N+1)}{(N+1)-j+1} \quad &- \quad \sum_{j=1}^{n^*-1} j \cdot \frac{(N+2)}{N-j+1} \\
&= \quad 1 + \sum_{j=1}^{n^*-1} (j+1) \cdot \frac{(N+1)}{N-j+1} - \sum_{j=1}^{n^*-1} j \cdot \frac{(N+2)}{N-j+1} \\
&= \quad 1 + \sum_{j=1}^{n^*-1} \underbrace{\frac{(j+1)\cdot(N+1) - j\cdot(N+2)}{N-j+1}}_{=1} \\
&= \quad n^* \,. \qquad (31)
\end{aligned}$$

Finally we get with Table 4, (30) and (31)

$$\begin{aligned}
(N+1)\cdot Op^*_{N+1} \quad &- \quad (N+2)\cdot Op^*_N \\
&= \quad \sum_{j=1}^{n^*} j \cdot \frac{(N+1)}{(N+1)-j+1} + (n^*+1)\cdot(N+1)\cdot p^*_{n^*+1}(N+1) \\
&- \quad \sum_{j=1}^{n^*-1} j \cdot \frac{(N+2)}{N-j+1} - n^* \cdot (N+2)\cdot p^*_{n^*}(N) \\
&= \quad n^* + n^* \cdot \left(-2 + \sum_{j=1}^{n^*-1} \frac{1}{N-j+1} \right) + (N+1)\cdot p^*_{n^*+1}(N+1) \\
&= \quad -n^* + n^* \cdot \sum_{j=1}^{n^*-1} \frac{1}{N-j+1} + (N+1)\cdot \left(1 - \sum_{j=1}^{n^*} \frac{1}{(N+1)-j+1} \right) \\
&= \quad (N-n^*+1) + n^* \cdot \sum_{j=1}^{n^*-1} \frac{1}{N-j+1} - \underbrace{\sum_{j=1}^{n^*} \frac{(N+1)}{(N+1)-j+1}}_{=1+\sum_{j=1}^{n^*-1} \frac{(N+1)}{N-j+1}} \\
&= \quad (N-n^*+1)\cdot \left(1 - \sum_{j=1}^{n^*-1} \frac{1}{N-j+1} \right) - 1 \\
&= \quad (N-n^*+1)\cdot \underbrace{\left(1 - \sum_{j=1}^{n^*} \frac{1}{N-j+1} \right)}_{\leq\ 0 \text{ from } (10)} \\
&\leq \quad 0\,,
\end{aligned}$$

which completes the proof. □

Up to now we have been considering the zero-sum game with payoff matrix A. If N increases, the reference time interval is getting longer and longer. However, from a practical point of view the reference time interval is constant. For that reason we start considering the zero-sum game with the same pure strategy sets $\Phi_{Op} = \{0, 1, \ldots, N\}$ and $\Phi_{Insp} = \{1, \ldots, N\}$ but now with the payoff matrix $\frac{1}{N+1}A$. This new game has the reference time interval 1 (for instance one year) and the beginning of the illegal action resp. the interim inspections take place at time points $0, 1/(N+1), \ldots, N/(N+1)$ resp. $1/(N+1), \ldots, N/(N+1)$. Since we have only multiplied the payoff's with a positive constant, this game possesses the same saddle point(s) like the original game (see, e.g., [Kar59]). Let $\tilde{n}(N)$ resp. $\tilde{Op}_N$ be the cutting edge resp. the value of the game for the zero-sum game with payoff matrix $\frac{1}{N+1}A$. Then it holds:

$$\tilde{n}(N) = \frac{1}{N+1}\, n^*(N) \qquad \text{and} \qquad \tilde{Op}_N = \frac{1}{N+1}\, Op^*_N\,.$$

We now want to investigate the asymptotic behavior of the cutting edge $\tilde{n}(N)$, the value of the game $\tilde{Op}_N$ and the saddle point strategies for the zero-sum game with payoff matrix $\frac{1}{N+1}A$. Let $s \in [0,1]$ be given. Then there exists a natural number $l(s,N) \in \{0, \ldots, N+1\}$ and $\delta(s,N) \in [0, 1/(N+1))$ with $s = \frac{l(s,N)}{N+1} + \delta(s,N)$. We define

$$Q^*_N(s) := \sum_{i=0}^{l(s,N)} q^*_i(N)\,. \tag{32}$$

The cumulative distribution function $Q^*_N(s)$ of $\mathbf{q}^*$ is the probability that in the zero-sum game with payoff matrix $\frac{1}{N+1}A$ the start of the illegal action is performed at time point s or earlier. The cumulative distribution function $P^*_N(t)$ of $\mathbf{p}^*$ can be defined in a similar way: For a given $t \in [0,1]$ there exists a natural number $l(t,N) \in \{0, \ldots, N+1\}$ and $\delta(t,N) \in [0, 1/(N+1))$ with $t = \frac{l(t,N)}{N+1} + \delta(t,N)$. We define

$$P^*_N(t) := \begin{cases} 0 & \text{for} \quad 0 \le t < \dfrac{1}{N+1} \\ \displaystyle\sum_{j=1}^{l(t,N)} p^*_j(N) & \text{for} \quad \dfrac{1}{N+1} \le t \le 1 \end{cases}\,. \tag{33}$$

$P^*_N(t)$ is the probability that in the zero-sum game with payoff matrix $\frac{1}{N+1}A$ the inspection is performed at time point t or earlier.

The next Theorem deals with the asymptotic behavior of these functions, the cutting edge and the value of the game.

Theorem 2. *Consider the zero-sum game with payoff matrix $\frac{1}{N+1}A$, where matrix A is given by (1). Then we obtain for the cutting edge $\tilde{n}(N)$ and the value of the game $\tilde{Op}_N$ the following asymptotic behavior*

$$\lim_{N\to\infty} \tilde{n}(N) = \lim_{N\to\infty} \frac{1}{N+1}\, n^*(N) = 1 - \frac{1}{e} \approx 0.632121$$

and

$$\lim_{N\to\infty} \tilde{Op}_N = \lim_{N\to\infty} \frac{1}{N+1}\, Op^*_N = \frac{1}{e} \approx 0.367879\,.$$

Furthermore we get

$$\lim_{N\to\infty} Q_N^*(s) = Q^*(s) := \begin{cases} \dfrac{1}{e}\,\dfrac{1}{1-s} & s \in [0, 1-1/e) \\ 1 & s \in [1-1/e, 1] \end{cases} \tag{34}$$

and

$$\lim_{N\to\infty} P_N^*(t) = P^*(t) = \begin{cases} (-1)\ln[(1-t)] & t \in [0, 1-1/e) \\ 1 & t \in [1-1/e, 1] \end{cases} . \tag{35}$$

Proof. In this proof we frequently use the so-called Sandwich Theorem. Let $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$ and $(c_k)_{k\in\mathbb{N}}$ be sequences of real numbers such that

$$a_k \le b_k \le c_k$$

at least for all $k \ge k_0$. If $\lim_{k\to\infty} a_k = \lim_{k\to\infty} c_k =: \alpha$, then $\lim_{k\to\infty} b_k = \alpha$.

From (21) we get

$$\left(1 - \frac{1}{e}\right)\frac{N}{N+1} < \frac{1}{N+1}\,n^*(N) < \left(1 - \frac{1}{e}\right) + \frac{1}{N+1}$$

and with the Sandwich Theorem

$$\lim_{N\to\infty} \frac{1}{N+1}\,n^*(N) = 1 - \frac{1}{e}\,,$$

as required. From (22) we have

$$\frac{(N - n^* + 1)}{N+1} < \frac{1}{N+1}\,Op_N^* < \frac{(N - n^* + 2)}{N+1}\,.$$

Using the first statement of this Theorem we have

$$\lim_{N\to\infty} \frac{(N - n^* + 1)}{N+1} = \lim_{N\to\infty} \frac{(N - n^* + 2)}{N+1} = \frac{1}{e}\,,$$

which shows with the Sandwich Theorem the relation

$$\lim_{N\to\infty} \frac{1}{N+1}\,Op_N^* = \frac{1}{e}\,.$$

We first show the asymptotic behavior $\lim_{N\to\infty} P_N(t) = P^*(t)$ for all $t \in [0,1]$. Let $t \in [0, 1-1/e)$. We define

$$N_1(t) := \frac{t+2}{(1 - \frac{1}{e}) - t}\,.$$

Then $N_1(t) > 0$ and we obtain for all $N \ge N_1(t)$

$$\left(\left(1 - \frac{1}{e}\right) - t\right)\cdot N \ge t + 2$$

and therewith

$$\left(1-\frac{1}{e}\right)\cdot N \geq t\cdot(N+1)+2 \geq \underbrace{(t-\delta(t,N))\cdot(N+1)}_{=\,l(t,N)}+2\,.$$

This leads with (21) to

$$n^*(N) > \left(1-\frac{1}{e}\right)\cdot N \geq l(t,N)+2 \quad \text{for all} \quad N \geq N_1(t)\,.$$

This condition ensures that in the following summations we only have to consider $p_j^*(N)$ from the first line of formula (12). Making use of definition (33) we first have for all N and all $t \geq \frac{1}{N+1}$

$$\sum_{j=1}^{l(t,N)} p_j^*(N) = P_N^*(t) \leq \sum_{j=1}^{l(t,N)+1} p_j^*(N) \tag{36}$$

and therefore for all $N \geq N_1(t)$

$$\sum_{j=1}^{l(t,N)} \frac{1}{N-j+1} = P_N^*(t) \leq \sum_{j=1}^{l(t,N)+1} \frac{1}{N-j+1}$$

and with (23)

$$\ln\left[\frac{N+1}{N-l(t,N)+1}\right] \leq P_N^*(t) \leq \ln\left[\frac{N}{N-l(t,N)-1}\right]\,,$$

which is equivalent to

$$\ln\left[\frac{1+1/N}{1-l(t,N)/N+1/N}\right] \leq P_N^*(t) \leq \ln\left[\frac{1}{1-l(t,N)/N}\right]\,.$$

Since $\lim_{N\to\infty} \frac{l(t,N)}{N} = t$ we get

$$\lim_{N\to\infty} \frac{1+1/N}{1-l(t,N)/N+1/N} = \lim_{N\to\infty} \frac{1}{1-l(t,N)/N} = \frac{1}{1-t}$$

and therefore with the Sandwich Theorem $\lim_{N\to\infty} P_N^*(t) = P^*(t) = (-1)\cdot\ln[\,(1-t)\,]$ for all $t \in (0, 1-1/e)$. Since $P_N^*(0) = P^*(0)$ equation (35) holds also for $t = 0$. In case of $t \in (1-1/e, 1]$ we define

$$N_2(t) := \frac{2}{t-\left(1-\frac{1}{e}\right)} - 1\,.$$

Then $N_2(t) > 0$ and we have for all $N \geq N_2(t)$

$$(N+1)\cdot\left(t-\left(1-\frac{1}{e}\right)\right) \geq 2\,,$$

which is equivalent to

$$(N+1)\cdot\left(t-\frac{1}{N+1}\right)\geq\left(1-\frac{1}{e}\right)\cdot(N+1)+1\,.$$

Since $\delta(t,N)<1/(N+1)$ we finally obtain with (21)

$$\underbrace{(N+1)\cdot(t-\delta(t,N))}_{=\,l(t,N)}>(N+1)\cdot\left(t-\frac{1}{N+1}\right)\geq\left(1-\frac{1}{e}\right)\cdot(N+1)+1>n^*(N)\,,$$

i.e., the condition

$$l(t,N)\geq n^*(N)+1\quad\text{for all}\quad N\geq N_2(t)\,.$$

Using (33) and (12) we have for all $N\geq N_2(t)$

$$1=\sum_{j=1}^{l(t,N)}p_j^*(N)=P_N^*(t)\leq\sum_{j=1}^{l(t,N)+1}p_j^*(N)=1\,.$$

If $\tilde{t}:=1-\frac{1}{e}$ we have for any N the representation $\tilde{t}=\frac{l(\tilde{t},N)}{N+1}+\delta(\tilde{t},N)$ with $l(\tilde{t},N)\in\{0,1,\ldots,N+1\}$ and $\delta(\tilde{t},N)\in[0,1/(N+1))$, which leads with (36) to

$$\begin{aligned}P_N^*\left(\tilde{t}-\delta(\tilde{t},N)\right)=P_N^*\left(\frac{l(\tilde{t},N)}{N+1}\right)=P_N^*(\tilde{t})\quad&\leq\quad P_N^*\left(\frac{l(\tilde{t},N)+1}{N+1}\right)\\&=\quad P_N^*\left(\tilde{t}+\frac{1}{N+1}-\delta(\tilde{t},N)\right)\,.\end{aligned}$$

Since $\lim_{N\to\infty}\delta(\tilde{t},N)=0$ and $\tilde{t}-\delta(\tilde{t},N)<\tilde{t}$ (because $\tilde{t}$ is an irrational number) we have

$$\begin{aligned}\lim_{N\to\infty}P_N^*\left(\tilde{t}-\delta(\tilde{t},N)\right)\;&=\;\lim_{N\to\infty}(-1)\cdot\ln[\,1-(\tilde{t}-\delta(\tilde{t},N))\,]=\lim_{N\to\infty}(-1)\cdot\ln\left[\frac{1}{e}+\delta(\tilde{t},N)\right]\\&=\;\ln[e]=1\,.\end{aligned}$$

Since $\tilde{t}+\frac{1}{N+1}-\delta(\tilde{t},N)>\tilde{t}$ we finally get

$$1=\lim_{N\to\infty}P_N^*\left(\tilde{t}-\delta(\tilde{t},N)\right)=P^*(\tilde{t})\leq 1\,.$$

Together we conclude $\lim_{N\to\infty}P_N^*(t)=P^*(t)$ for all $t\in[0,1]$, which has to be proven. We now show the asymptotic behavior $\lim_{N\to\infty}Q_N(t)=Q^*(s)$ for all $s\in[0,1]$. Let $s\in[0,1-1/e)$. With definition (32) we get for all $N\geq N_1(s)$

$$\sum_{i=0}^{l(s,N)}q_i^*(N)=Q_N^*(s)\leq\sum_{i=0}^{l(s,N)+1}q_i^*(N)$$

and with (14)

$$(N-n^*+1)\cdot\frac{1}{N-l(s,N)}=Q_N^*(s)\leq(N-n^*+1)\cdot\frac{1}{N-l(s,N)-1}\,.$$

Using the first statement of this Theorem and $\lim_{N\to\infty} \frac{l(s,N)}{N} = s$ we get

$$\lim_{N\to\infty} \frac{N-n^*+1}{N} \cdot \frac{1}{1-\frac{l(s,N)}{N}} = \lim_{N\to\infty} \frac{N-n^*+1}{N} \cdot \frac{1}{1-\frac{l(s,N)}{N}-\frac{1}{N}} = \frac{1}{e}\,\frac{1}{1-s}\,.$$

The Sandwich Theorem implies $\lim_{N\to\infty} Q_N^*(s) = \frac{1}{e}\cdot\frac{1}{1-s}$ for all $s \in [0, 1-1/e)$. In case of $s \in (1-1/e, 1]$ we obtain for all $N \geq N_2(s)$

$$1 = \sum_{i=0}^{l(s,N)} q_i^*(N) = Q_N^*(s) \leq \sum_{i=0}^{l(s,N)+1} q_i^*(N) = 1\,.$$

In case of $\tilde{s} := 1 - \frac{1}{e}$ we get

$$\begin{aligned} Q_N^*\left(\tilde{s} - \delta(\tilde{s},N)\right) = Q_N^*\left(\frac{l(\tilde{s},N)}{N+1}\right) = Q_N^*(\tilde{s}) \quad &\leq \quad Q_N^*\left(\frac{l(\tilde{s},N)+1}{N+1}\right) \\ &= \quad Q_N^*\left(\tilde{s} + \frac{1}{N+1} - \delta(\tilde{s},N)\right)\,. \end{aligned}$$

Since $\lim_{N\to\infty} \delta(\tilde{s},N) = 0$ and $\tilde{s} - \delta(\tilde{s},N) < \tilde{s}$ (because $\tilde{s}$ is an irrational number) we have

$$\lim_{N\to\infty} Q_N^*\left(\tilde{s} - \delta(\tilde{s},N)\right) = \lim_{N\to\infty} \frac{1}{e}\cdot\frac{1}{\frac{1}{e}+\delta(\tilde{s},N)} = 1\,.$$

Since $\tilde{s} + \frac{1}{N+1} - \delta(\tilde{s},N) > \tilde{s}$ we finally get

$$1 = \lim_{N\to\infty} Q_N^*\left(\tilde{s} - \delta(\tilde{s},N)\right) = Q^*(\tilde{s}) \leq 1\,.$$

Together we have $\lim_{N\to\infty} Q_N^*(s) = Q^*(s)$ for all $s \in [0,1]$, which finishes the proof. □

Let us remark that if N is sufficiently large only 2/3 of time points are really used for inspection.

We mentioned in the beginning that we consider in this paper only the illegal game, i.e., the game, where legal behavior of the operator is a priori excluded. Including legal behavior and introducing losses and gains for legal behavior and for performing an illegal action will lead to a formal condition for legal behavior of the operator. This condition then allows to determine the size of the sanctions for detected illegal behavior such that the operator is induced to legal behavior.

4 The continuous time inspection game

In Table 2 we see that the inspector can indeed improve his strategic advantage in the normalized zero-sum game simply by increasing the number of inspection opportunities from say 3 to 7, while still making only one interim visit per year. One might ask, what the expected detection time would be if, rather than being allowed to inspect every month, he could come at the end of every week or every day or, completely unannounced, any time he wished. This leads directly to games with infinitely many pure strategies (see, e.g., [Kar59]).

Representing the inspection year by the interval $[0, 1]$, the operator starts his illegal action at time $s \in [0, 1]$ and the inspector chooses his interim inspection at time $t \in [0, 1]$. The operator's payoff, in analogy to (1), is given by the so-called payoff kernel

$$\tilde{A}(s, t) := \begin{cases} t - s & 0 \leq s < t \\ 1 - s & t \leq s < (\leq) \, 1 \end{cases} . \tag{37}$$

This game can be understood as the continuous version of the zero-sum discrete time inspection game with the payoff matrix A given by (1). A mixed strategy for the operator resp. inspector is a probability distribution on [0,1]. Let $Q(s)$ be the probability of diversion occurring at time s or earlier and let $P(t)$ be the probability of an inspection having taken place at time t or earlier. Using Lebesque-Stieltjes integrals (see [CvB00]), we define, in analogy to (3), the expected payoff to the operator by

$$Op(Q, P) := \int_0^1 \int_0^1 \tilde{A}(s, t) \, dQ(s) \, dP(t) \, .$$

A mixed strategy combination (Q^*, P^*) constitutes a saddle point if and only if

$$Op(Q, P^*) \leq Op(Q^*, P^*) \leq Op(Q^*, P) \quad \text{for all} \quad Q \quad \text{and} \quad P \, .$$

Infinite games with discontinuous payoff kernels, such as this one, may have no saddle point at all, see, e.g., [Owe68]. Nevertheless it would be surprising if a limiting case of the discrete time inspection game would have no solution. Fortunately, it can be shown, that for the game discussed here, at least one saddle point exists. This is formulated in

Theorem 3. *The zero-sum game over the unit square with payoff kernel in (37) has the following solution. The operator chooses his start of the illegal action s according to the distribution function $Q^*(s)$ given by (34), while the inspector chooses the inspection time t according to the distribution function $P^*(t)$ given by (35). The value of the game is $1/e$.*

Proof: The proof can be found in [AC96] or [AvSZ02]. □

Comparing these results with those from Theorem 2 we get the main result of this contribution: The saddle point strategies of the zero-sum discrete time inspection game with payoff matrix A can be seen – for large N – as an approximation of the saddle point strategies of the continuous time inspection game with payoff kernel (37). This is not an obvious result. If the game in this section had a continuous payoff kernel over $[0, 1] \times [0, 1]$ this asymptotic behavior were obvious. However, the game considered here possesses a discontinuous payoff kernel. This asymptotic relation is remarkable; it may be guessed but has to be proven.

Although the solution of the continuous time inspection game is more manageable then the solution of the discrete time inspection game, one has always to solve the discrete time inspection game, if one has to consider a practical situation with a finite number of time points for interim inspections: Even if the number of time points for interim inspections gets large, the distribution functions of the continuous time inspection game can simply not be used as those of the discrete time inspection game.

5 Acknowledgement

The author would like to thank Rudolf Avenhaus, Universität der Bundeswehr München, for helpful suggestions and active support. The Institut für Technik Intelligenter Systeme e.V. (ITIS) an der Universität der Bundeswehr München has provided the possibility to work on the problems discussed here.

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